In presenting the dissertation as a partial fulfillment of the requirements for an advanced degree from the Georgia Institute of Technology, I agree that the Library of the Institute shall make it available for inspection and circulation in accordance with its regulations governing materials of this type. I agree that permission to copy from, or to publish from, this dissertation may be granted by the professor under whose direction it was written, or, in his absence, by the Dean of the Graduate Division when such copying or publication is solely for scholarly purposes and does not involve potential financial gain. It is understood that any copying from, or publication of, this dissertation which involves potential financial gain will not be allowed without written permission.

7/25/68

# ROTATIONALLY SYMMETRIC VIBRATIONS
# OF ORTHOTROPIC LAYERED
# CYLINDRICAL SHELLS

A THESIS

Presented to

The Faculty of the Graduate Division

by

Teh-Min Hsu

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

in the School of Engineering Science and Mechanics

Georgia Institute of Technology

May, 1969

ROTATIONALLY SYMMETRIC VIBRATIONS

OF ORTHOTROPIC LAYERED

CYLINDRICAL SHELLS

Approved:

Chairman

Date approved by Chairman: April 28, 1969

This thesis is dedicated to

my parents and Mei.

## ACKNOWLEDGMENTS

The author wishes to express his sincere appreciation to his thesis advisor, Dr. James T. S. Wang, for his helpful assistance and guidance in completing the thesis, and to Dr. M. E. Raville for his arrangement of financial assistance during the author's period of study at the Georgia Institute of Technology.

The author also wishes to express his gratitude to the members of the reading committee, Dr. E. R. Wood and Dr. David J. McGill, for their valuable suggestions.

Appreciation is also extended to the staff of the Rich Electronic Computer Division for the use of the Burroughs 5500 digital computer and related equipment.

# TABLE OF CONTENTS

## LIST OF TABLES

## LIST OF ILLUSTRATIONS

## LIST OF SYMBOLS

All symbols are defined in the text where they first appear and the major symbols are listed below:

| | |
|---|---|
| $\lfloor \bar{D} \rceil$ | Coefficient matrix |
| $h$ | Thickness of cylindrical shell |
| $\ell$ | Length of cylindrical shell |
| $m$ | Mode number |
| $N$ | Total number of Layers of Laminated shell |
| $\{\bar{p}\}$ | External loading matrix |
| $R^j$ | Mean radius of the jth layer of layered cylindrical shell |
| $u,v,w$ | Displacement components in longitudinal, circumferential, and normal directions respectively |
| $x,y,z$ | Cylindrical coordinates |
| $P^j,Q^j$ | Longitudinal and normal components of surface loads applied to the jth contact surface |
| $u_o,v_o$ | Displacement components of the middle surface of the shell in x and y directions respectively |
| $X^+,Y^+$ | Longitudinal and circumferential components of outer surface loads |
| $X^-,Y^-$ | Longitudinal and circumferential components of inner surface loads |
| $X^j,Z^j$ | Longitudinal and normal components of surface loads applied to the jth contact surface |
| $M_x,M_y,M_{xy},M_{yx}$ | Stress couples |
| $T_x,T_y,T_{xy},T_{yx}$ | Stress resultants |

| | |
|---|---|
| $a_{ik}, B_{ik}, \Omega$ | Material elastic constants |
| $A_{in}, \bar{a}_{in}, g_{in}, T_n,$ $I_{kin}, J_{kin}, J^*_{in},$ $L_{kin}, \alpha^*_{in}, \bar{\alpha}_{kin},$ $\zeta^*_{in}, \zeta_{kin}$ | Constants which are functions of material elastic constants, dimensions of the shell and an integer n |
| $b_n, d_n, t_{in}, r_{in}$ | Coefficients of Fourier series |
| $C_k, \bar{c}_k, D_k, F_k, f_k,$ $\bar{f}_k, g_k, G_k, m_k, M_k,$ $q_k, s_k, \eta^*_k, \eta_k, E^*$ | Constants which are functions of material elastic constants and dimensions and natural frequency of the shell |
| $K_i$ | Constants of integration |
| $[\bar{D}_{ik}]$ | Submatrix of matrix $[\bar{D}]$ |
| $[\bar{F}_{ik}], [\bar{H}_{ik}], [\bar{\Lambda}_{ik}], [\Gamma_{ik}]$ | Submatrices of matrix $[\bar{D}_{ik}]$ |
| $\varphi$, $\psi$ | Functions which characterize the variation of the transverse shear stresses in x-z and y-z planes respectively |
| $\sigma_x, \sigma_y, \sigma_z$ | Normal stress components |
| $\tau_{xy}, \tau_{yz}, \tau_{xz}$ | Shear stress components |
| $\epsilon_x, \epsilon_y, \epsilon_z$ | Normal strain components |
| $\gamma_{xy}, \gamma_{yz}, \gamma_{xz}$ | Shear strain components |
| $\rho$ | Material mass density |
| $\bar{\lambda}$ | Frequency parameter |
| $\omega$ | Natural circular frequency |
| $\alpha_k, \beta_k, \lambda_k, \xi_k$ | Parameters used in the characteristic equation (3.25) |

Superscripts

| | |
|---|---|
| j | Quantities which relate to the jth-layered shell |

## SUMMARY

This dissertation is concerned with the theoretical investigation of rotationally symmetric vibration of orthotropic layered cylindrical shells of finite length, without the hypothesis of non-deformable normals. A closed form analytical solution is obtained. Each layered shell may have different thickness and/or be made of different orthotropic (or isotropic) materials.

In order to be able to estimate the stresses developed in the bond material, each layer is treated separately. With shear deformations and longitudinal inertia included, stress resultants and stress couples in terms of displacement components and shear functions are formulated and the general governing differential equations of motion for a general layer are derived.

By using the generalized Fourier series to express the unknown stresses between each pair of adjacent layers, and requiring the satisfaction of both compatibility conditions at interacting surfaces and homogeneous boundary conditions along edges of each individual layered shell, the rotationally symmetric vibration problem of an N-layered orthotropic cylindrical shell is solved exactly. Solutions for one and two layered cylindrical shells with fixed-fixed edges are presented in detail. Numerical results of shells made of barite and topaz are obtained for illustrative purposes.

# CHAPTER I

## INTRODUCTION

The demand for increasing structural efficiency of many advanced vehicle designs has resulted in the use of a layered shell construction. For example, an atmospheric entry vehicle may consist of a metallic inner structure, to which is bonded a cover layer of heat shield material. Since shear stresses at the interfaces between the shell layers, which will be carried by the bonding material, are of considerable interest, it is necessary to consider shear deformation in the shell theory. A new theory without the hypothesis of nondeformable normals which permits estimates of stresses in the bond material will be developed.

The existing general theories of anisotropic layered shells were developed on the basis of the Kirchhoff theory of isotropic shells by taking into account the anisotropic material properties. Many analyses were made by considering all layers as one equivalent layer with effective stiffnesses. The derivations of the equations of motion, kinematic relations, and compatibility and boundary conditions were the same as the existing theories for uniform shells.

For the most general anisotropic elastic body, the generalized Hooke's Law contains 21 independent elastic constants. The generalized Hooke's Law reads

$$\epsilon_k = a_{ki}\sigma_i \qquad i,\ k = 1,2,3,\ldots,6,$$

where $\varepsilon_k$ and $\sigma_i$ are strain and stress components, respectively, and $a_{ki}$ are elastic constants.

If there is a plane of elastic symmetry, then the elastic constants reduce to 13, i.e., $a_{k4}$, $a_{k5}$, for $k = 1,2,3$, and $a_{46}$, $a_{56}$ are zero. For an orthotropic case, there are only 9 independent elastic constants.

The original investigations on the theory of anisotropic shells, carried out by Shtaerman [1]* in 1924, deal with the problem of the theory of symmetrically loaded orthotropic shells of revolution. The membrane theory of anisotropic shells has been considered by Ambartsumian [2] and Dong [3]. The basic equations of the theory of orthotropic shells were derived by Mushtari [4], [5] in 1938. The theory of anisotropic layered cylindrical shells was first investigated by Ambartsumian [6] in 1953. Eason [7] considered radial vibration for the case of an infinitely long anisotropic cylinder. Das [8] derived Donnell and Vlasov types of equations for an orthotropic single-layered cylindrical shell according to classical thin shell theory with in-plane inertia neglected. General solutions for a simply supported shell are presented, but no numerical results are given. Mirsky [9], [10], applied the Frobenius series method to solve the problem of axisymmetric vibration of infinitely long orthotropic cylinders with one layer. Kalnins [11], by defining the equivalent density of a layered shell as $\rho = \sum_{j=1}^{M} \rho^j (z^{j+1} - z^j)$,

*The numbers in brackets denote the correspondingly numbered reference in the literature cited.

generalized the theory of rotationally symmetric thin elastic shells and applied the results to both isotropic and orthotropic layered cylindrical shells of infinite length. Although Ahmed [12] has considered the thick shell problem, his investigation was limited to the case of infinitely long cylinders; hence the boundary conditions have no effect on its axisymmetric vibration; furthermore, only plane strain was considered. Thus, all theoretical studies for vibration of anisotropic cylinders are based upon either the Kirchhoff hypothesis or upon considering only the radial vibration of infinitely long cylinders.

As indicated by Ambartsumian [13], the hypothesis of nondeformable normals, while acceptable for isotropic shells, is often quite unacceptable for anisotropic shells, even if the anisotropic shell is relatively thin ($h/R \ll 1$). The Kirchhoff hypothesis, i.e. the one based upon the hypothesis of nondeformable normals, is quite indifferent to relations of the type $G_{k3}/E_{ki}$, $E_{33}/E_{ki}$ ($G_{k3}$ transverse moduli of shear, $E_{33}$ transverse modulus of elasticity, $E_{ki}$ moduli of elasticity in the direction of middle surface), and thus it often contains substantial error. Therefore, a more exact theory of shells which are constructed of anisotropic material may be developed only by abandoning the hypothesis of nondeformable normals.

In the present analysis, the hypothesis of nondeformable normals will not be invoked but the inextensibility of the thickness together with finite length cylinders are considered; as a result, one obtains a "two and a half dimensional elasticity theory" [14].

Since the cylinders are finite, this will necessitate considering boundary conditions at both ends of the shell. It is assumed that the bond between layers of the shell is sufficiently thin so that the geometry of the shell system is not altered and the bond inertia can be neglected. In order to be able to determine the effect of the flexibility of the bond, each layered shell is considered separately. Stress resultants and stress couples in terms of displacement components and shear functions are formulated and the governing differential equations for each orthotropic cylindrical layered shell are derived. The theoretical results are extended to the analysis of N-layered cylindrical shells. The general solution presented is immediately applicable to the case of orthotropic (or isotropic) layered cylindrical shells of finite length with any homogeneous boundary conditions and an arbitrary number of layers. The frequency equation, whose form depends on numerical values of elastic constants and the geometric dimensions of the shell, of a single-layered shell with fixed-fixed edges is considered in detail. Numerical results of several eigenfrequencies and corresponding mode shapes for both one-layered shells and two-layered shells constructed of barite and topaz are presented for shells with different lengths and thicknesses.

These results may also serve as a prelude for subsequent analysis of the dynamic response of the shell structures to various loads and to asymmetric vibrations of orthotropic layered shells.

# CHAPTER II

## DERIVATIONS OF GOVERNING DIFFERENTIAL EQUATIONS

The hypothesis of nondeformable normals, while acceptable for isotropic shells, is often quite unacceptable for anisotropic shells as indicated in [13], even if the anisotropic shell is relatively thin. In addition, the hypothesis will not give an adequate estimate of shear stresses between layers. It is, therefore, necessary to derive a new theory of shells without using the hypothesis of nondeformable normals.

### Basic Assumptions

The basic assumptions used in this analysis are as follows:

1. All layers of the laminated shell remain elastic in the presence of deformation, i.e. they obey the generalized Hooke's Law.

2. The shell is transversely inextensible, i.e. the distance along the normal between two points of the shell is unchanged before and after deformation.

3. No slippage takes place between layers.

4. The bond between layers is sufficiently thin that the geometry of the shell system is not altered, and bond inertia can be neglected.

5. The variation of transverse shearing stresses $\tau_{xz}$ and $\tau_{yz}$ is represented in the following forms for each layer:

$$\tau_{xz} = f_1(z)\varphi(x,y) + \frac{z}{h}(X^+ + X^-) + \frac{X^+ - X^-}{2}, \quad (2.1a)$$

$$\tau_{yz} = f_2(z)\psi(x,y) + \frac{z}{h}(Y^+ + Y^-) + \frac{Y^+ - Y^-}{2}, \quad (2.1b)$$

where x, y, z are cylindrical coordinates as shown in Figure 1; $X^+(x,y)$ $X^-(x,y),\ldots, Y^-(x,y)$ are tangential components of the intensity of surface loads, applied to the outer surfaces of the shell; $\varphi(x,y)$, $\psi(x,y)$ are the desired functions characterizing the variation of transverse shear; $f_1(z)$ and $f_2(z)$ are functions which represent the variation of transverse shear stresses $\tau_{xz}$ and $\tau_{yz}$, with $f_1(\pm \frac{1}{2} h)$ and $f_2(\pm \frac{1}{2} h) = 0$, where h is the thickness of the layer. In this analysis, the functions $f_1(z)$ and $f_2(z)$ are chosen to be

$$f_1(z) = f_2(z) = \frac{1}{4} h^2 - z^2 . \quad (2.2)$$

## Stress-strain Relationships

As shown in the Appendix A, the stress-strain relationships for an orthotropic elastic material may be written as

$$\sigma_x = B_{11}\varepsilon_x + B_{12}\varepsilon_y, \quad (2.3)$$

$$\sigma_y = B_{12}\varepsilon_x + B_{22}\varepsilon_y,$$

$$\sigma_z = B_{13}\varepsilon_x + B_{23}\varepsilon_y,$$

$$\tau_{xz} = B_{55}\gamma_{xz},$$

$$\tau_{yz} = B_{44}\gamma_{yz},$$

$$\tau_{xy} = B_{66}\gamma_{xy},$$

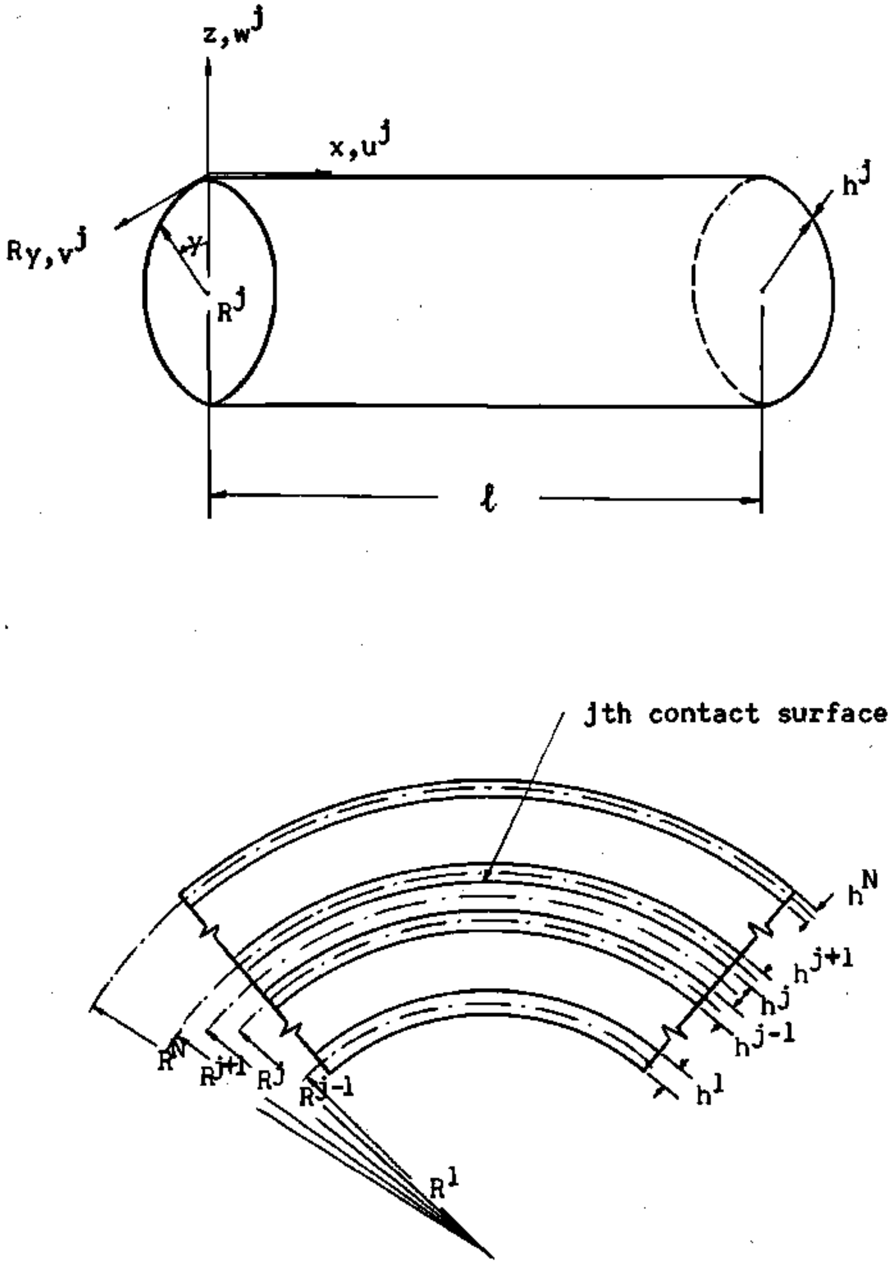



Figure 1. Coordinate System for a Layered Cylindrical Shell.

where $\sigma_x$, $\sigma_y$,...,$\tau_{xy}$ are stress components; $\varepsilon_x$, $\varepsilon_y$,...,$\gamma_{xy}$ are strain components, and $B_{in}$ are elastic constants defined in Appendix A.

## Strain-displacement Relationships

The strain-displacement relationships for a cylindrical shell are as follows:

$$\varepsilon_x = \frac{\partial u}{\partial x} , \tag{2.4}$$

$$\varepsilon_y = \frac{1}{R+z} \left(\frac{\partial v}{\partial y} + w\right) ,$$

$$\varepsilon_z = \frac{\partial w}{\partial z} ,$$

$$\gamma_{xy} = \frac{1}{R+z} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} ,$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} - \frac{v}{R+z} + \frac{1}{R+z} \frac{\partial w}{\partial y} ,$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} ,$$

where u, v and w are displacement components in longitudinal, circumferential, and normal directions, respectively, as shown in Figure 1; and R is the mean radius of the layer.

## Stress Resultants and Stress Couples

By the assumption of transverse inextensibility, from the third equation of equations (2.4), one obtains

$$\varepsilon_z = \frac{\partial w}{\partial z} = 0$$

or

$$w = w(x, y) . \tag{2.5}$$

The substitution of equation (2.1) into the 4th and 5th equations of equations (A.1) results in the expressions

$$\gamma_{xz} = a_{55}\left[f_1\varphi + \frac{z}{h}(X^+ + X^-) + \frac{X^+ - X^-}{2}\right], \qquad (2.6)$$

$$\gamma_{yz} = a_{44}\left[f_2\psi + \frac{z}{h}(Y^+ + Y^-) + \frac{Y^+ - Y^-}{2}\right]. \qquad (2.7)$$

Substituting equations (2.5), (2.6) and (2.7) into the 5th and 6th equations of equation (2.4), and integrating with respect to the normal coordinate, z, yields the following expressions for longitudinal and circumferential displacements:

$$u(x,y,z) = u_o(x,y) - z\frac{\partial w(x,y)}{\partial x} + f_1^*(z)\varphi(x,y) \qquad (2.8)$$

$$+ f_3(z)\,X^+(x,y) + f_4(z)\,X^-(x,y),$$

and

$$v(x,y,z) = v_o(x,y)(1 + \frac{z}{R}) + f_2^*(z)\,\psi(x,y) + \frac{\partial w(x,y)}{\partial y} \qquad (2.9)$$

$$+ f_5(z)\,Y^+(x,y) + f_6(z)Y^-(x,y)\ ,$$

where $u_o$, $v_o$ are the tangential displacements of the middle surface of the layer in the x and y directions respectively, and where

$$f_1^*(z) = a_{55}\int_0^z f_1(\zeta)d\zeta\ , \qquad (2.10)$$

$$f_2^*(z) = a_{44}(R + z)\int_0^z \frac{f_2(\zeta)}{R+\zeta}\,d\zeta\ ,$$

$$f_3(z) = \frac{a_{55}}{2}\cdot z(1 + \frac{z}{h}),$$

$$f_4(z) = \frac{a_{55}}{2} \cdot z(-1 + \frac{z}{h}) ,$$

$$f_5(z) = \frac{a_{44}}{2} (R + z)[\frac{z}{h} + (\frac{1}{2} - \frac{R}{h})\ln(R+z)] ,$$

$$f_6(z) = \frac{a_{44}}{2} (R + z)[\frac{z}{h} - (\frac{1}{2} + \frac{R}{h})\ln(R+z)] .$$

From equation (2.4), the strain-displacement relations then have the forms

$$\varepsilon_x = \frac{\partial u_o}{\partial x} - z\frac{\partial^2 w}{\partial x^2} + f_1^* \frac{\partial \varphi}{\partial x} + f_3 \frac{\partial X^+}{\partial x} + f_4 \frac{\partial X^-}{\partial x} , \tag{2.11}$$

$$\varepsilon_y = \frac{1}{R}\frac{\partial v_o}{\partial y} + \frac{f_2^*}{R+z}\frac{\partial \psi}{\partial y} + \frac{1}{R+z}\frac{\partial^2 w}{\partial y^2} + \frac{w}{R+z} + \frac{f_5}{R+z}\frac{\partial Y^+}{\partial y} + \frac{f_6}{R+z}\frac{\partial Y^-}{\partial z} ,$$

$$\varepsilon_z = \frac{\partial w}{\partial z} = 0 ,$$

$$\gamma_{xy} = \frac{1}{R+z}\frac{\partial u_o}{\partial y} + \frac{R}{R+z}\frac{\partial^2 w}{\partial x \partial y} + \frac{f_1^*}{R+z}\frac{\partial \varphi}{\partial y} + (1 + \frac{z}{R})\frac{\partial v_o}{\partial x} + f_2^*\frac{\partial \psi}{\partial x}$$

$$+ \frac{f_3}{R+z}\frac{\partial X^+}{\partial y} + \frac{f_4}{R+z}\frac{\partial X^-}{\partial y} + f_4\frac{\partial Y^+}{\partial x} + f_6\frac{\partial Y^-}{\partial x} ,$$

and $\gamma_{xz}$, $\gamma_{yz}$ are indicated in equations (2.6) and (2.7).

Substituting equation (2.11) into equation (2.3), one can express the stress components in terms of displacement components and normal coordinate as follows:

$$\sigma_x = B_{11}\frac{\partial u_o}{\partial x} + \frac{B_{12}}{R}\frac{\partial v_o}{\partial y} + \frac{B_{12}}{R+z} \cdot w - B_{11} \cdot z\frac{\partial^2 w}{\partial x^2} \tag{2.12}$$

$$+ \frac{B_{12}}{R+z}\frac{\partial^2 w}{\partial y^2} + \frac{B_{12}f_2^*}{R+z}\frac{\partial \psi}{\partial y} + B_{11}f_1^*\frac{\partial \varphi}{\partial x} + B_{11}f_3\frac{\partial X^+}{\partial x}$$

$$+ B_{11}f_4 \frac{\partial X^-}{\partial x} + \frac{B_{12}f_5}{R+z} \frac{\partial Y^+}{\partial y} + \frac{B_{12}f_6}{R+z} \frac{\partial Y^-}{\partial y} ,$$

$$\sigma_y = B_{12} \frac{\partial u_o}{\partial x} + \frac{B_{22}}{R} \frac{\partial v_o}{\partial y} + \frac{B_{22}}{R+z} \cdot w - B_{12} z \frac{\partial^2 w}{\partial x^2} + \frac{B_{22}}{R+z} \frac{\partial^2 w}{\partial y^2}$$

$$+ \frac{B_{22}f_2^*}{R+z} \frac{\partial \psi}{\partial y} + B_{12}f_1^* \frac{\partial \varphi}{\partial x} + B_{12}f_3 \frac{\partial X^+}{\partial x} + B_{12}f_4 \frac{\partial X^-}{\partial x}$$

$$+ \frac{B_{22}f_5}{R+z} \frac{\partial Y^+}{\partial y} + \frac{B_{22}f_6}{R+z} \frac{\partial Y^-}{\partial y} ,$$

$$\sigma_z = - \frac{a_{13}}{a_{33}} \sigma_x - \frac{a_{23}}{a_{33}} \sigma_y ,$$

$$\tau_{xy} = \frac{B_{66}}{R+z} \frac{\partial u_o}{\partial y} + \frac{B_{66}(R+z)}{R} \frac{\partial v_o}{\partial x} + B_{66}\left(\frac{R}{R+z}\right) \frac{\partial^2 w}{\partial x \partial y} + \frac{B_{66}f_1^*}{R+z} \frac{\partial \varphi}{\partial y}$$

$$+ B_{66}f_2^* \frac{\partial \psi}{\partial x} + \frac{B_{66}f_3}{R+z} \frac{\partial X^+}{\partial y} + \frac{B_{66}f_4}{R+z} \frac{\partial X^-}{\partial y} + \frac{B_{66}f_5}{R+z} \frac{\partial Y^+}{\partial x}$$

$$+ \frac{B_{66}f_6}{R+z} \frac{\partial Y^-}{\partial x} ,$$

$$\tau_{yz} = f_2 \psi + \frac{z}{h} (Y^+ + Y^-) + \frac{Y^+ - Y^-}{2} ,$$

$$\tau_{xz} = f_1 \varphi + \frac{z}{h} (X^+ + X^-) + \frac{X^+ - X^-}{2} .$$

Stress resultants $\{T_1, T_2, T_{12}, T_{21}, N_1, N_2\}$ and stress couples $\{M_1, M_2, M_{12}, M_{21}\}$ can be formulated in terms of displacement of middle surface of the shell by the integration of equation (2.12) over the shell thickness as follows:

$$\{ T_x, T_{xy}, N_x \} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \{\sigma_x, \tau_{xy}, \tau_{xz}\} \left(1 + \frac{\zeta}{R_2}\right) d\zeta , \qquad (2.13)$$

$$\{T_y,\ T_{yx},\ N_y\} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \{\sigma_y,\ \tau_{yx},\ \tau_{yz}\}(1 + \frac{\zeta}{R_1})\, d\zeta\ ,$$

$$\{M_x,\ M_{xy}\} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \{\sigma_x,\ \tau_{xy}\} \cdot \zeta(1 + \frac{\zeta}{R_2})\, d\zeta\ ,$$

and

$$\{M_y,\ M_{yx}\} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \{\sigma_y,\ \tau_{yx}\} \cdot \zeta(1 + \frac{\zeta}{R_1})\, d\zeta\ .$$

After carrying through the integrations, one obtains

$$\begin{aligned} T_x = {} & B_{11} h \frac{\partial u_o}{\partial x} + \frac{h}{R} B_{12} \frac{\partial v_o}{\partial y} + \frac{h}{R} B_{12} \cdot w - \frac{B_{11} h^3}{12R} \frac{\partial^2 w}{\partial x^2} + \frac{B_{12} h}{R} \frac{\partial^2 w}{\partial y^2} \qquad (2.14) \\ & + \frac{c_1 a_{44} B_{12}}{R} \frac{\partial \psi}{\partial y} + \frac{a_{55} h^5 B_{11}}{60\, R} \frac{\partial \varphi}{\partial x} + \frac{B_{11} h^3 a_{55}}{24\, R} (1 + \frac{R}{h}) \frac{\partial X^+}{\partial x} \\ & + \frac{B_{11} a_{55} h^3}{24\, R} (\frac{R}{h} - 1) \frac{\partial X^-}{\partial x} + \frac{B_{12} a_{44}}{R} [\frac{h^2}{12} - c_{13}(\frac{R}{h} - \frac{1}{2})] \frac{\partial Y^+}{\partial y} \\ & + \frac{B_{12} a_{44}}{R} [\frac{h^2}{12} - c_{13}(\frac{R}{h} + \frac{1}{2})] \frac{\partial Y^-}{\partial y}\ , \end{aligned}$$

$$\begin{aligned} T_y = {} & B_{12} h \frac{\partial u_o}{\partial x} + \frac{B_{22} h}{R} \frac{\partial v_o}{\partial y} + B_{22} \ln(\frac{2R + h}{2R - h}) \cdot w + B_{22} \ln(\frac{2R + h}{2R - h}) \frac{\partial^2 w}{\partial y^2} \\ & + c_2 B_{22} \frac{\partial \psi}{\partial y} + \frac{B_{12} a_{55}}{24} (\frac{\partial X^+}{\partial x} + \frac{\partial X^-}{\partial x}) + a_{44} B_{22} c_{17} \frac{\partial Y^+}{\partial y} \\ & + a_{44} B_{22} c_{18} \frac{\partial Y^-}{\partial y}\ , \end{aligned}$$

$$T_{xy} = \frac{B_{66}h}{R}\frac{\partial u_o}{\partial y} + \frac{B_{66}}{R^2}(R^2h + \frac{h^3}{12})\frac{\partial v_o}{\partial x} + B_{66}h\frac{\partial^2 w}{\partial x \partial y}$$

$$+ \frac{c_3a_{44}B_{66}}{R}\frac{\partial \psi}{\partial x} + \frac{B_{66}h^2a_{55}}{24\,R}(\frac{\partial X^+}{\partial y} + \frac{\partial X^-}{\partial y}) + a_{44}B_{66}c_{19}\frac{\partial Y^+}{\partial x}$$

$$+ a_{44}B_{66}c_{20}\frac{\partial Y^-}{\partial x} ,$$

$$T_{yx} = B_{66}\ln(\frac{2R+h}{2R-h})\frac{\partial u_o}{\partial y} + B_{66}h\frac{\partial v_o}{\partial x} + B_{66}R\ln(\frac{2R+h}{2R-h})\frac{\partial^2 w}{\partial x \partial y}$$

$$+ c_4a_{55}B_{66}\frac{\partial \varphi}{\partial y} + c_5a_{44}B_{66}\frac{\partial \psi}{\partial x} + \frac{a_{55}B_{66}c_{12}}{2}(1 - \frac{R}{h})\frac{\partial X^+}{\partial y}$$

$$- \frac{c_{12}a_{55}B_{66}}{2}(1 + \frac{R}{h})\frac{\partial X^-}{\partial y} + a_{44}B_{66}[\frac{h^2}{12} - c_{13}(\frac{R}{h} - \frac{1}{2})]\frac{\partial Y^+}{\partial x}$$

$$+ a_{44}B_{66}[\frac{h^2}{12} - c_{13}(\frac{R}{h} + \frac{1}{2})]\frac{\partial Y^-}{\partial x} ,$$

$$N_x = \frac{h^3}{6}\varphi + \frac{h}{2}(\frac{h}{6R} + 1)X^+ + \frac{h}{2}(\frac{h}{6R} - 1)X^- ,$$

$$N_y = \frac{h^3}{6}\psi + \frac{h}{2}(Y^+ - Y^-) ,$$

$$M_x = \frac{h^3}{12R}(B_{11}\frac{\partial u_o}{\partial x} + \frac{B_{12}}{R}\frac{\partial v_o}{\partial y} - R\,B_{11}\frac{\partial^2 w}{\partial x^2}) + \frac{c_6B_{12}}{R}\frac{\partial \psi}{\partial y}$$

$$+ \frac{a_{55}h^5B_{11}}{60}\frac{\partial \varphi}{\partial x} + \frac{a_{55}B_{11}h^3}{8}(\frac{h}{20R} + \frac{1}{3})\frac{\partial X^+}{\partial x}$$

$$+ \frac{a_{55}B_{11}h^3}{8}(\frac{h}{20R} - \frac{1}{3})\frac{\partial X^-}{\partial x} + [\frac{h^2}{12} - c_{14}(\frac{1}{h} - \frac{1}{2R}) + c_{13}(\frac{R}{h} - \frac{1}{2})]\frac{\partial Y^+}{\partial y}$$

$$+ [\frac{h^2}{12} - c_{14}(\frac{1}{h} + \frac{1}{2R}) + c_{13}(\frac{R}{h} + \frac{1}{2})]\frac{\partial Y^-}{\partial y} ,$$

$$M_y = c_{12}B_{22}w - \frac{B_{12}h^3}{12}\frac{\partial^2 w}{\partial x^2} + c_{12}B_{22}\frac{\partial^2 w}{\partial y^2} + c_7a_{44}B_{22}\frac{\partial \psi}{\partial y}$$

$$+ \frac{a_{55}h^5B_{12}}{60}\frac{\partial \varphi}{\partial x} + \frac{a_{55}B_{12}h^3}{24}(\frac{\partial X^+}{\partial x} - \frac{\partial X^-}{\partial x}) + a_{44}B_{22}[\frac{h^2}{12} - c_{15}(\frac{R}{h} - \frac{1}{2})]\cdot$$

$$\frac{\partial Y^+}{\partial y} + a_{44}B_{22}[\frac{h^2}{12} - c_{15}(\frac{R}{h} + \frac{1}{2})]\frac{\partial Y^-}{\partial y} ,$$

$$M_{xy} = \frac{B_{66}h^3}{6R}\frac{\partial v_o}{\partial x} + \frac{a_{55}B_{66}h^5}{60}\frac{\partial \varpi}{\partial y} + c_8 a_{44}B_{66}\frac{\partial \psi}{\partial x} + \frac{B_{66}a_{55}h^3}{24R} \cdot$$

$$\cdot (\frac{\partial X^+}{\partial y} - \frac{\partial X^-}{\partial y}) + \frac{a_{44}B_{66}}{R}[\frac{R^2h^2}{12} + \frac{h^4}{80} - c_{16}(\frac{R}{h} - \frac{1}{2})]\frac{\partial Y^+}{\partial x}$$

$$+ \frac{a_{44}B_{66}}{R}[\frac{R^2h^2}{12} + \frac{h^4}{80} - c_{16}(\frac{R}{h} + \frac{1}{2})]\frac{\partial Y^-}{\partial x} ,$$

$$M_{yx} = c_{12}B_{66}\frac{\partial u_o}{\partial y} + \frac{B_{66}h^3}{12R}\frac{\partial v_o}{\partial x} + c_9 B_{66}\frac{\partial^2 w}{\partial x \partial y} + c_{10}a_{55}B_{66}\frac{\partial \varpi}{\partial y}$$

$$+ c_{11}a_{44}B_{66}\frac{\partial \psi}{\partial x} + \frac{a_{55}B_{66}}{2}[\frac{h^2}{12} + c_{12}R\ (\frac{R}{h} - 1)]\frac{\partial X^+}{\partial y}$$

$$+ \frac{a_{55}B_{66}}{2}[\frac{h^2}{12} + c_{12}R(\frac{R}{h} + 1)]\frac{\partial X^-}{\partial y} + a_{44}B_{66}R[\frac{h^2}{12} + c_{13}(\frac{R}{h} - \frac{1}{2})$$

$$- c_{14}(\frac{1}{h} - \frac{1}{2R})]\frac{\partial Y^+}{\partial x} + a_{44}B_{66}R[\frac{h^2}{12} + c_{13}(\frac{R}{h} + \frac{1}{2}) - c_{14}(\frac{1}{h} + \frac{1}{2R})]\frac{\partial Y^-}{\partial x} ,$$

where $c_j$, for $j = 1,2,\ldots, 20$ are constants and they are

$$c_1 = \frac{Rh^3}{24} + R(R^2 - \frac{h^2}{4})[h \ln R - R \ln(\frac{2R+h}{2R-h}) - \frac{h}{2}\ln(R^2 - \frac{h^2}{4}) + h] \quad (2.15)$$

$$-(R^2 - \frac{h^2}{4})[\frac{Rh}{2} - \frac{1}{2}(R^2 - \frac{h^2}{4})\ln(\frac{2R+h}{2R-h})] ,$$

$$c_2 = -\frac{h^3}{24} + (R - \frac{h^2}{4})[h \ln R - R \ln(\frac{2R+h}{2R-h}) - \frac{h}{2}\ln(R^2 - \frac{h^2}{4}) + h],$$

$$c_3 = \frac{1}{8}R^2h^3 + R(R^2 - \frac{h^2}{4})[\frac{Rh}{2} + Rh \ln R - \frac{Rh}{2}\ln(R^2 - \frac{h^2}{4})$$

$$- \frac{4R^2 + h^2}{8}\ln(\frac{2R+h}{2R-h})] - \frac{1}{80}h^5 - (R^2 - \frac{h^2}{4})[\frac{h-R}{6}(R + \frac{h}{2})^2 \cdot$$

$$\ln\left(R+\frac{h}{2}\right)+\frac{R+h}{6}\left(R-\frac{h}{2}\right)^2\ln\left(R-\frac{h}{2}\right)+\frac{R^2h}{6}-\frac{h^3}{36}$$

$$-\frac{h^3}{12}\ln R\Big],$$

$$c_4=-\frac{1}{3}R^2h+\frac{2}{9}h^3+\left(\frac{R^2}{3}-\frac{h^2}{4}\right)R\ln\left(\frac{2R+h}{2R-h}\right),$$

$$c_5=\frac{R^3h}{2}-\frac{Rh^3}{12}+\left(R^2-\frac{h^2}{4}\right)\Big[Rh\ln R-\frac{Rh}{2}\ln\left(R^2-\frac{h^2}{4}\right)$$

$$-\frac{4R^2+h^2}{8}\ln\left(\frac{2R+h}{2R-h}\right)\Big],$$

$$c_6=\frac{R^2h^3}{12}-R\left(R^2-\frac{h^2}{4}\right)\left[\frac{Rh}{2}-\frac{1}{2}\left(R^2-\frac{h^2}{4}\right)\ln\left(\frac{2R+h}{2R-h}\right)\right]$$

$$-\frac{h^5}{160}+\left(R^2-\frac{h^2}{4}\right)\frac{h^3}{12}\ln R-\left(R^2-\frac{h^2}{4}\right)\Big[\left(R+\frac{h}{2}\right)$$

$$\cdot\frac{R^2-\frac{Rh}{2}+\frac{h^2}{4}}{3}\ln\left(R+\frac{h}{2}\right)-\left(R-\frac{h}{2}\right)\frac{R^2+\frac{Rh}{2}+\frac{h^2}{4}}{3}\ln\left(R-\frac{h}{2}\right)$$

$$-\frac{1}{3}R^2h-\frac{h^3}{36}\Big],$$

$$c_7=\frac{Rh^3}{12}-\left(R^2-\frac{h^2}{4}\right)\left[\frac{Rh}{2}-\frac{1}{2}\left(R^2-\frac{h^2}{4}\right)\ln\left(\frac{2R+h}{2R-h}\right)\right],$$

$$c_8=R\left(\frac{R^2h^3}{12}-\frac{h^5}{80}\right)-R\left(R^2-\frac{h^2}{4}\right)\Big[\frac{h-R}{6}\left(R+\frac{h}{2}\right)^2\ln\left(R+\frac{h}{2}\right)$$

$$+\frac{R+h}{6}\left(R-\frac{h}{2}\right)^2\ln\left(R-\frac{h}{2}\right)+\frac{1}{6}R^2h-\frac{h^3}{36}-\frac{h^3}{12}\ln R\Big]$$

$$+\frac{Rh^5}{160}+\frac{Rh^3}{12}\left(R^2-\frac{h}{4}\right)\ln R+\left(R^2-\frac{h^2}{4}\right)\Big[\frac{Rh}{12}\left(R^2+\frac{h^2}{6}\right)$$

$$-\left(R+\frac{h}{2}\right)^2\frac{R^2-Rh+\frac{3}{4}h^2}{12}\ln\left(R+\frac{h}{2}\right)+\left(R-\frac{h}{2}\right)^2\frac{R^2+Rh+\frac{3}{4}h^2}{12}$$

$$\cdot\ln\left(R-\frac{h}{2}\right)\Big],$$

$$c_9 = Rh - R^2 \ln \left(\frac{2R + h}{2R - h}\right) ,$$

$$c_{10} = \frac{1}{3} R^3 h - \frac{2}{9} Rh^3 - R^2 \left(\frac{R^2}{3} - \frac{h^2}{4}\right) \ln \left(\frac{2R + h}{2R - h}\right) ,$$

$$c_{11} = \frac{R^2 h^3}{12} - \frac{h^5}{80} - \left(R^2 - \frac{h^2}{4}\right)\left[\frac{h - R}{6}\left(R + \frac{h}{2}\right)^2 \ln \left(R + \frac{h}{2}\right)\right.$$

$$\left. + \frac{R + h}{6} \left(R - \frac{h}{2}\right)^2 \ln \left(R - \frac{h}{2}\right) + \frac{1}{6} R^2 h - \frac{h^3}{36} - \frac{h^3}{12} \ln R\right] ,$$

$$c_{12} = h - R \ln \left(\frac{2R + h}{2R - h}\right) = \frac{c_9}{R} ,$$

$$c_{13} = -\frac{Rh}{2} + \frac{4R^2 + h^2}{8} \ln \left(\frac{2R + h}{2R - h}\right) + \frac{Rh}{2} \ln \left(R^2 - \frac{h^2}{4}\right) ,$$

$$c_{14} = \frac{R}{3} \left(R^2 + \frac{3}{4} h^2\right) \ln\left(\frac{2R + h}{2R - h}\right) + \frac{h}{6} \left(3R^2 + \frac{1}{4} h^2\right) \ln\left(R^2 - \frac{h}{4}\right)$$

$$- \frac{h}{9} \left(3R^2 + \frac{h^2}{4}\right) ,$$

$$c_{15} = \frac{Rh}{2} - \frac{1}{2} \left(R^2 - \frac{h^2}{4}\right) \ln \left(\frac{2R + h}{2R - h}\right) ,$$

$$c_{16} = -\frac{Rh}{16} \left(4R^2 + h^2\right) + \frac{R^4 + \frac{3}{2} R^2 h^2 + \frac{1}{16} h^4}{4} \ln\left(\frac{2R + h}{2R - h}\right) - c_{14} R$$

$$+ \frac{2R^3 h + \frac{1}{2} Rh^3}{4} \ln \left(R^2 - \frac{h^2}{4}\right) ,$$

$$c_{17} = \left[c_{12} - \frac{h}{2} \ln \left(R^2 - \frac{h^2}{4}\right)\right]\left(\frac{R}{h} + \frac{1}{2}\right) ,$$

$$c_{18} = \left[c_{12} - \frac{h}{2} \ln \left(R^2 - \frac{h^2}{4}\right)\right]\left(\frac{R}{h} - \frac{1}{2}\right) ,$$

$$c_{19} = \frac{1}{R} \left[\frac{Rh^2}{6} + \frac{h^4}{80} - c_{14}\left(\frac{R}{h} - \frac{1}{2}\right)\right] ,$$

$$c_{20} = \frac{1}{R}\left[\frac{Rh^2}{6} + \frac{h^4}{80} - c_{14}\left(\frac{R}{h} + \frac{1}{2}\right)\right].$$

## Governing Differential Equations of Motion

The equations of motion of a general shell element can be obtained by including the inertia terms in the equilibrium equations which are available in any standard text book on shell theories such as [25]. When the coordinate system coincides with the lines of principal curvature, the equations of motion become

$$\frac{1}{A_1A_2}\left[\frac{\partial A_2T_x}{\partial x} + \frac{\partial A_1T_{yx}}{\partial y} + \frac{\partial A_1}{\partial y}T_{xy} - \frac{\partial A_2}{\partial x}T_y\right] + \frac{N_x}{R_1} + q_x = \rho h\frac{\partial^2 u}{\partial t^2}, \quad (2.16)$$

$$\frac{1}{A_1A_2}\left[\frac{\partial A_2T_{xy}}{\partial x} + \frac{\partial A_1T_y}{\partial y} + \frac{\partial A_2}{\partial x}T_{yx} - \frac{\partial A_1}{\partial y}T_x\right] + \frac{N_y}{R_2} + q_y = \rho h\frac{\partial^2 v}{\partial t^2},$$

$$\frac{1}{A_1A_2}\left[\frac{\partial A_2N_x}{\partial x} + \frac{\partial A_1N_y}{\partial y}\right] - \frac{T_x}{R_1} - \frac{T_y}{R_2} + q_n = \rho h\frac{\partial^2 w}{\partial t^2},$$

$$\frac{1}{A_1A_2}\left[\frac{\partial A_2M_x}{\partial x} + \frac{\partial A_1M_{yx}}{\partial y} + \frac{\partial A_1}{\partial y}M_{xy} - \frac{\partial A_2}{\partial x}M_y\right] - N_x = 0,$$

$$\frac{1}{A_1A_2}\left[\frac{\partial A_2M_{xy}}{\partial x} + \frac{\partial A_1M_y}{\partial y} + \frac{\partial A_2}{\partial x}M_{yx} - \frac{\partial A_1}{\partial y}M_x\right] - N_y = 0,$$

$$T_{xy} - T_{yx} + \frac{M_{xy}}{R_1} - \frac{M_{yx}}{R_2} = 0,$$

where $A_1$ and $A_2$ are Lamé parameters, $R_1$ and $R_2$ are radii of principal curvature, and $q_x$, $q_y$ and $q_n$ are surface loading functions. For a cylindrical coordinate system, $A_1 = 1$, $A_2 = R$, $R_1 = \infty$ and $R_2 = R$, where R is the radius of the middle surface of a cylindrical shell.

Substitution of the set of equations (2.14) into the equations of motion (2.16) yields the following governing differential equations of motion for a general orthotropic single-layered cylindrical shell:

$$
\begin{aligned}
& B_{11}h\,\frac{\partial^2 u_o}{\partial x^2} + \frac{B_{66}}{R}\ln\left(\frac{2R+h}{2R-h}\right)\frac{\partial^2 u_o}{\partial y^2} + \frac{h}{R}\,(B_{12}+B_{66})\,\frac{\partial^2 v_o}{\partial x\partial y} \\
& + \frac{h}{R}\,B_{12}\,\frac{\partial w}{\partial x} - \frac{B_{11}h^3}{12R}\,\frac{\partial^3 w}{\partial x^3} + \ln\left(\frac{2R+h}{2R-h}\right)B_{66}\,\frac{\partial^3 w}{\partial x\partial y^2} + \frac{1}{R}\,(c_1 a_{44}B_{12} \\
& + c_5 a_{44}B_{66})\,\frac{\partial^2 \varphi}{\partial x\partial y} + \frac{B_{12}h}{R}\,\frac{\partial^3 w}{\partial x\partial y^2} + \frac{a_{55}h^5}{60R}\,B_{11}\,\frac{\partial^2 \varphi}{\partial x^2} + \frac{c_4 a_{55}B_{66}}{R}\,\frac{\partial^2 \varphi}{\partial y^2} \\
& + \frac{a_{55}B_{11}h^2}{24}\left[1+\frac{h}{R}\right]\frac{\partial^2 \chi^+}{\partial x^2} + \frac{a_{55}B_{11}h^2}{24}\left(1-\frac{h}{R}\right)\frac{\partial^2 \chi^-}{\partial x^2} \\
& + \frac{a_{55}B_{66}c_{12}}{2h}\left(-1+\frac{h}{R}\right)\frac{\partial^2 \chi^+}{\partial y^2} - \frac{a_{55}B_{66}c_{12}}{2h}\left(1+\frac{h}{R}\right)\frac{\partial^2 \chi^-}{\partial y^2} \\
& + (B_{12}+B_{66})a_{44}\left[\frac{h^2}{12R} - c_{13}\left(\frac{1}{h}-\frac{1}{2R}\right)\right]\frac{\partial^2 \gamma^+}{\partial x\partial y} \\
& + (B_{12}+B_{66})a_{44}\left[\frac{h^2}{12R} - c_{13}\left(\frac{1}{h}+\frac{1}{2R}\right)\right]\frac{\partial^2 \gamma^-}{\partial x\partial y} + q_x = \rho h\,\frac{\partial^2 u}{\partial t^2}\,,
\end{aligned}
\tag{2.17a}
$$

$$
\begin{aligned}
& \frac{h}{R}\,(B_{66}+B_{12})\,\frac{\partial^2 u_o}{\partial x\partial y} + \frac{B_{66}h}{R^2}\left(R^2+\frac{h^2}{12}\right)\frac{\partial^2 v_o}{\partial x^2} + \frac{B_{22}h}{R^2}\,\frac{\partial^2 v_o}{\partial y^2} \\
& + \frac{B_{22}}{R}\ln\left(\frac{2R+h}{2R-h}\right)\frac{\partial w}{\partial y} + B_{66}h\,\frac{\partial^3 w}{\partial x^2\partial y} + \frac{B_{22}}{R}\ln\left(\frac{2R+h}{2R-h}\right)\frac{\partial^3 w}{\partial y^3} \\
& + \frac{h^3}{6R}\,\varphi + \frac{c_3 B_{66}}{R}\,\frac{\partial^2 \varphi}{\partial x^2} + \frac{c_2 B_{22}}{R}\,\frac{\partial^2 \varphi}{\partial y^2} + \frac{h}{2R}\,(\gamma^+ - \gamma^-) \\
& + \frac{a_{55}h^2}{24R}\,(B_{12}+B_{66})\left(\frac{\partial^2 \chi^+}{\partial x\partial y} + \frac{\partial^2 \chi^-}{\partial x\partial y}\right) + \frac{a_{44}B_{66}}{R}\left[\frac{Rh^2}{6}\right. \\
& \left. - c_{14}\left(\frac{R}{h}-\frac{1}{2}\right)\right]\frac{\partial^2 \gamma^+}{\partial x^2} + \frac{a_{44}B_{66}}{R}\left[\frac{Rh^2}{6} - c_{14}\left(\frac{R}{h}+\frac{1}{2}\right)\right]\frac{\partial^2 \gamma^-}{\partial x^2}
\end{aligned}
\tag{2.17b}
$$

$$+ a_{44}B_{22}[c_{12} - \frac{h}{2} \ln (R^2 - \frac{h^2}{4})] \left\{ (\frac{1}{h} - \frac{1}{2R}) \frac{\partial^2 \gamma^+}{\partial y^2} \right.$$

$$\left. + (\frac{1}{h} + \frac{1}{2R}) \frac{\partial^2 \gamma^-}{\partial y^2} \right\} + q_y = \rho h \frac{\partial^2 v}{\partial t^2} ,$$

$$\frac{B_{12}h}{R} \frac{\partial u_o}{\partial x} + \frac{B_{22}h}{R^2} \frac{\partial v_o}{\partial y} + \frac{B_{22}}{R} \ln (\frac{2R + h}{2R - h}) w + \frac{B_{22}}{h} \ln (\frac{2R + h}{2R - h}) \frac{\partial^2 w}{\partial y^2} \quad (2.17c)$$

$$- \frac{h^3}{6} \frac{\partial \varphi}{\partial x} + (\frac{c_2}{R} B_{22} - \frac{h^3}{6R}) \frac{\partial \varphi}{\partial y} - (\frac{h^2}{12R} + \frac{h}{2} + \frac{a_{55}B_{12}h^2}{24R}) \frac{\partial \chi^+}{\partial x}$$

$$- (\frac{h^2}{12R} - \frac{h}{2} + \frac{a_{55}B_{12}h^2}{24R}) \frac{\partial \chi^-}{\partial x} - \left\{ B_{22}a_{44}(1 - \frac{h}{2R}) [\frac{c_{12}}{h} \right.$$

$$\left. - \frac{1}{2} \ln (R^2 - \frac{h^2}{4})] + \frac{h}{2R} \right\} \frac{\partial \gamma^+}{\partial y} - \left\{ B_{22}a_{44}(1 + \frac{h}{2R}) [\frac{c_{12}}{h} \right.$$

$$\left. - \frac{1}{2} \ln (R^2 - \frac{h^2}{4})] - \frac{h}{2R} \right\} \frac{\partial \gamma^-}{\partial y} - q_n = - \rho h \frac{\partial^2 w}{\partial t^2} ,$$

$$\frac{B_{11}h^3}{12R} \frac{\partial^2 u_o}{\partial x^2} + \frac{c_{12}B_{66}}{R} \frac{\partial^2 u_o}{\partial y^2} + \frac{h^3}{12R^2} (B_{12} + B_{66}) \frac{\partial^2 v_o}{\partial x \partial y} \quad (2.17d)$$

$$- \frac{B_{11}h^3}{12} \frac{\partial^3 w}{\partial x^3} + \frac{c_9 B_{66}}{R} \frac{\partial^3 w}{\partial x \partial y^2} - \frac{h^3 \varphi}{6} + \frac{a_{55}h^5 B_{11}}{60} \frac{\partial^2 \varphi}{\partial x^2}$$

$$+ \frac{c_{10}a_{55}B_{66}}{R} \frac{\partial^2 \varphi}{\partial y^2} + \frac{1}{R} (c_6 B_{12} + c_{11}a_{44}B_{66}) \frac{\partial^2 \varphi}{\partial x \partial y} - h(\frac{h}{2R} + \frac{1}{2}) \chi^+$$

$$- h(\frac{h}{12R} - \frac{1}{2}) \chi^- + \frac{a_{55}B_{11}h^3}{8} (\frac{h}{20R} + \frac{1}{3}) \frac{\partial^2 \chi^+}{\partial x^2}$$

$$+ \frac{a_{55}B_{11}h^3}{8} (\frac{h}{20R} - \frac{1}{3}) \frac{\partial^2 \chi^-}{\partial x^2} + \frac{a_{55}B_{66}}{2} [\frac{h^2}{12R} + c_{12}(\frac{R}{h} - 1)] \frac{\partial^2 \chi^+}{\partial y^2}$$

$$+ \frac{a_{55}B_{66}}{2} [\frac{h^2}{12R} + c_{12} (\frac{R}{h} + 1)] \frac{\partial^2 \chi^-}{\partial y^2} + a_{44}(B_{12} + B_{66}) \cdot$$

$$\cdot \left[\frac{h^2}{12} - \frac{c_{14}}{h}\left(1 - \frac{h}{2R}\right) + c_{13}\left(\frac{R}{h} - \frac{1}{2}\right)\right] \frac{\partial^2 \gamma^+}{\partial x \partial y} + a_{44}(B_{12} + B_{66}) \cdot$$

$$\cdot \left[\frac{h^2}{12} - \frac{c_{14}}{h}\left(1 + \frac{h}{2R}\right) + c_{13}\left(\frac{R}{h} + \frac{1}{2}\right)\right] \frac{\partial^2 \gamma^-}{\partial x \partial y} = 0 \ ,$$

$$\frac{B_{66}h^3}{6R} \frac{\partial^2 v_o}{\partial x^2} + \frac{c_{12}B_{22}}{R} \frac{\partial w}{\partial y} - \frac{B_{12}h^3}{12R} \frac{\partial^3 w}{\partial x^2 \partial y} + \frac{c_{12}B_{22}}{R} \frac{\partial^3 w}{\partial y^3} \tag{2.17e}$$

$$+ \frac{a_{55}h^5}{60R}(B_{66} + B_{12}) \frac{\partial^2 \varphi}{\partial x \partial y} - \frac{h^3}{6}\psi + \frac{c_8 a_{44} B_{66}}{R} \frac{\partial^2 \psi}{\partial x^2} + \frac{c_7 a_{44} B_{22}}{R} \frac{\partial^2 \psi}{\partial y^2}$$

$$+ \frac{h}{2}(\gamma^+ - \gamma^-) + \frac{a_{55}h^3}{24R}(B_{12} + B_{66})\left(\frac{\partial^2 \chi^+}{\partial x \partial y} - \frac{\partial^2 \chi^-}{\partial x \partial y}\right)$$

$$+ \frac{B_{66}a_{44}}{R}\left[\frac{R^2h^2}{12} + \frac{h^4}{80} - c_{16}\left(\frac{R}{h} - \frac{1}{2}\right)\right] \frac{\partial^2 \gamma^+}{\partial x^2}$$

$$+ \frac{B_{66}a_{44}}{R}\left[\frac{R^2h^2}{12} + \frac{h^4}{80} - c_{16}\left(\frac{R}{h} + \frac{1}{2}\right)\right] \frac{\partial^2 \gamma^-}{\partial x^2}$$

$$+ B_{22}a_{44}\left[\frac{h^2}{12R} - \frac{c_{15}}{h}\left(1 - \frac{h}{2R}\right)\right] \frac{\partial^2 \gamma^+}{\partial y^2}$$

$$+ B_{22}a_{44}\left[\frac{h^2}{12R} - \frac{c_{15}}{h}\left(1 + \frac{h}{2R}\right)\right] \frac{\partial^2 \gamma^-}{\partial y^2} = 0 \ .$$

The last equation of equations (2.16) is satisfied identically.

By using the assumption listed in the first section of this chapter, stress resultants and stress couples in terms of displacement components and shear functions are obtained and the general governing differential equations of motion for orthotropic single-layered cylindrical shell have now been formulated.

# CHAPTER III

## ROTATIONALLY SYMMETRIC VIBRATION OF A GENERAL LAYER OF A LAYERED CYLINDRICAL SHELL

The proposed analysis for a general cylindrical shell having multiple layers will treat each layer separately. The interaction between layers will be coupled by requiring compatible deformation and stresses. In order to investigate the rotationally symmetric vibration problem of an N-layered orthotropic cylindrical shell, the rotationally symmetric vibration problem of a single-layered orthotropic cylindrical shell must be formulated and studied first.

### Differential Equations of Motion of a General jth-layered Shell

For the case of rotationally symmetric vibration, the functions $u_o$, $w$, $\varphi$, $X^+$ and $X^-$ are functions of the longitudinal coordinate x and time only, while $v_o$, $\psi$, $Y^+$, and $Y^-$ are zero. Thus, the governing differential equations of motion (2.17) for a jth-layered orthotropic cylindrical shell reduce to the expressions

$$C_1^j \frac{\partial^2 u_o^j}{\partial x^2} + C_2^j \frac{\partial w^j}{\partial x} + C_3^j \frac{\partial^3 w^j}{\partial x^3} + C_4^j \frac{\partial^2 \varphi^j}{\partial x^2} + C_5^j \frac{\partial^2 X^j}{\partial x^2} \tag{3.1}$$

$$+ C_6^j \frac{\partial^2 X^{j-1}}{\partial x^2} - \rho^j h^j \frac{\partial^2 u_o^j}{\partial t^2} = 0 ,$$

$$D_1^j \frac{\partial u_o^j}{\partial x} + D_2^j w^j + D_3^j \frac{\partial \varphi^j}{\partial x} + D_4^j \frac{\partial X^j}{\partial x} + D_5^j \frac{\partial X^{j-1}}{\partial x} \tag{3.2}$$

$$+ \rho^j h^j \frac{\partial^2 w^j}{\partial t^2} + Z^j - Z^{j-1} = 0 ,$$

$$F_1^j \frac{\partial^2 u_o^j}{\partial x^2} + F_2^j \frac{\partial^3 w^j}{\partial x^3} + F_3^j \varphi^j + F_4^j \frac{\partial^2 \varphi^j}{\partial x^2} + F_5^j X^j \tag{3.3}$$

$$+ F_6^j X^{j-1} + F_7^j \frac{\partial^2 X^j}{\partial x^2} + F_8^j \frac{\partial^2 X^{j-1}}{\partial x^2} = 0 ,$$

where $X^j$, $Z^j$ and $X^{j-1}$, $Z^{j-1}$ are longitudinal and normal components of the surface load intensity, applied to the outer and the inner surfaces of the jth layer of a layered cylindrical shell respectively. For the free vibration case, $X^0$, $X^N$, $Z^0$, $Z^N$ are zero, where $j = 0$ and $j = N$ correspond to the inner-most and the outer-most surfaces of the layered cylindrical shell. $C_k^j$, $D_k^j$, and $F_k^j$ are constants whose values depend on the elastic properties of the material and the dimensions of the shell structure. They are

$$C_1^j = B_{11}^j h^j \ ; \quad C_2^j = \frac{h^j}{R^j} B_{12}^j \ ; \tag{3.4}$$

$$C_3^j = - \frac{B_{11}^j (h^j)^3}{12R^j} \ ; \quad C_4^j = \frac{a_{55}^j (h^j)^5}{60R^j} B_{11}^j \ ;$$

$$C_5^j = \frac{a_{55}^j B_{11}^j (h^j)^2}{24} \left(1 + \frac{h^j}{R^j}\right) ;$$

$$C_6^j = \frac{a_{55}^j B_{11}^j (h^j)^2}{24} \left(1 - \frac{h^j}{R^j}\right) ;$$

$$D_1^j = C_2^j = \frac{B_{12}^j h^j}{R^j} \ ;$$

$$D_2^j = \frac{B_{22}^j}{R^j} \ln \left(\frac{2R^j + h^j}{2R^j - h^j}\right) ;$$

$$D_3^j = -\frac{(h^j)^3}{6} ;$$

$$D_4^j = -\frac{B_{12}^j a_{55}^j (h^j)^2}{24R^j} - \frac{(h^j)^2}{12R^j} - \frac{h^j}{2} ;$$

$$D_5^j = -\frac{B_{12}^j a_{55}^j (h^j)^2}{24R^j} - \frac{(h^j)^2}{12R^j} + \frac{h^j}{2} ;$$

$$F_1^j = \frac{B_{11}^j (h^j)^3}{12R^j} = -C_3^j ; \quad F_2^j = -\frac{B_{11}^j (h^j)^3}{12} = C_3^j R^j ;$$

$$F_3^j = D_3^j = -\frac{(h^j)^3}{6} ; \quad F_4^j = \frac{a_{55}^j B_{11}^j (h^j)^5}{60} = C_4^j R^j ;$$

$$F_5^j = -\frac{(h^j)^2}{12R^j} - \frac{h^j}{2} ; \quad F_6^j = -\frac{(h^j)^2}{12R^j} + \frac{h^j}{2} ;$$

$$F_7^j = a_{55}^j B_{11}^j (h^j)^3 \left(\frac{h^j}{160R^j} + \frac{1}{24}\right) ;$$

$$F_8^j = a_{55}^j B_{11}^j (h^j)^3 \left(\frac{h^j}{160R^j} - \frac{1}{24}\right) ,$$

where the superscripts j stand for the jth layer of a layered shell and $a_{ik}$ are elastic constants defined in Appendix A.

For the free vibration of a jth-layered shell, one may assume that the solutions of partial differential equations (3.1) to (3.3) have the forms

$$u_o^j = U^j(x)e^{i\omega t}, \tag{3.5}$$

$$w = W^j(x)e^{i\omega t},$$

$$\Phi = \Phi^j(x)e^{i\omega t},$$

$$X^j = P^j(x)e^{i\omega t},$$

$$Z^j = Q^j(x)e^{i\omega t},$$

where $\omega$ is the circular frequency and $i = \sqrt{-1}$.

Substitution of equations (3.5) into the governing differential equations (3.1), (3.2), and (3.3) yields the results

$$C_1 \frac{d^2U}{dx^2} + C_2 \frac{dW}{dx} + C_3 \frac{d^3W}{dx^3} + C_4 \frac{d^2\Phi}{dx^2} + \rho h\omega^2 U = -C_5 \frac{d^2P^j}{dx^2} - C_6 \frac{d^2P^{j-1}}{dx^2}, \tag{3.6}$$

$$D_1 \frac{dU}{dx} + D_2 W + D_3 \frac{d\Phi}{dx} - \rho h\omega^2 W = -D_4 \frac{dP^j}{dx} - D_5 \frac{dP^{j-1}}{dx} - Q^j + Q^{j-1}, \tag{3.7}$$

$$F_1 \frac{d^2U}{dx^2} + F_2 \frac{d^3W}{dx^3} + F_3\Phi + F_4 \frac{d^2\Phi}{dx^2} = -F_5 P^j - F_6 P^{j-1} - F_7 \frac{d^2P^j}{dx^2} - F_8 \frac{d^2P^{j-1}}{dx^2}, \tag{3.8}$$

where the superscripts j of the coefficients and displacement components have been dropped to simplify the derivation. The superscripts j of the loading terms stand for the jth contact surface.

The elimination of $\frac{d^2\Phi}{dx^2}$ from equations (3.6) and (3.8) yields the equation

$$\Phi = f_1 U + f_2 \frac{d^2U}{dx^2} + f_3 \frac{dW}{dx} + f_4 P^j + f_5 P^{j-1} + f_6 \frac{d^2P^j}{dx^2} + f_7 \frac{d^2P^{j-1}}{dx^2} \qquad (3.9)$$

where

$$f_1 = B_{11} \bar{\lambda}^2 h \, F_4/f_0 \,, \qquad (3.10)$$

$$f_2 = (F_4 C_1 - F_1 C_4)/f_0 \,,$$

$$f_3 = F_4 C_2/f_0 \,,$$

$$f_4 = -F_5 C_4/f_0 \,,$$

$$f_5 = -F_6 C_4/f_0 \,,$$

$$f_6 = -(F_7 C_4 - F_4 C_5)/f_0$$

$$f_7 = -(F_8 C_4 - F_4 C_6)/f_0 \,,$$

and where

$$f_0 = F_3 C_4 \,, \qquad (3.10a)$$

$$\bar{\lambda}^2 = \rho\omega^2/B_{11} \,.$$

From equation (3.7), one obtains

$$\frac{d\Phi}{dx} = -\frac{1}{D_3}\left\{ D_1 \frac{dU}{dx} + D_2 W - \rho h \omega^2 W + D_4 \frac{dP^j}{dx} + D_5 \frac{dP^{j-1}}{dx} + Q^j - Q^{j-1} \right\}. \quad (3.11)$$

When equation (3.9) is differentiated and subsequently equated to equation (3.11), the result is

$$g_1 \frac{dU}{dx} + g_2 \frac{d^3U}{dx^3} + g_3 W + g_4 \frac{d^2W}{dx^2} + g_5 \frac{dP^j}{dx} + g_6 \frac{dP^{j-1}}{dx} + g_7 \frac{d^3P^j}{dx^3} \quad (3.12)$$

$$+ g_8 \frac{d^3P^{j-1}}{dx^3} + g_9 (Q^j - Q^{j-1}) = 0$$

where

$$g_1 = f_1 + D_1/D_3 \; ; \qquad g_2 = f_2 \; ; \qquad (3.13)$$

$$g_3 = (D_2 - \bar{\lambda}^2 B_{11} h)/D_3 \; ; \quad g_4 = f_3 \; ;$$

$$g_5 = f_5 + D_4/D_3 \; ; \qquad g_6 = f_6 + D_5/D_3 \; ;$$

$$g_7 = f_7 \; ; \quad g_8 = f_8 \; ; \quad g_9 = 1/D_3 \; .$$

The substitution of equation (3.11) into equation (3.6) results in

$$G_1 U + G_2 \frac{d^2U}{dx^2} + G_3 \frac{dW}{dx} + G_4 \frac{d^3W}{dx^3} + G_5 \frac{d^2P^j}{dx^2} + G_6 \frac{d^2P^{j-1}}{dx^2} \quad (3.14)$$

$$+ G_7 \left( \frac{dQ^j}{dx} - \frac{dQ^{j-1}}{dx} \right) = 0$$

where

$$G_1 = \bar{\lambda}^2 B_{11} h \; ; \qquad G_2 = C_1 - C_4 D_1/D_3 \; ; \qquad (3.15)$$

$$G_3 = C_2 - C_4 g_3 \; ; \qquad G_4 = C_3 \; ;$$

$$G_5 = C_5 - C_4D_4/D_3 \; ; \qquad G_6 = C_6 - C_4D_5/D_3 \; ;$$

$$G_7 = - C_4/D_3 \; .$$

Differentiation of equation (3.14) and substitution of the resulting $\frac{d^3U}{dx^3}$ into equation (3.12) gives

$$M_1 \frac{dU}{dx} + M_2W + M_3 \frac{d^2W}{dx^2} + M_4 \frac{d^4W}{dx^4} + M_5 \frac{dP^j}{dx} + M_6 \frac{dP^{j-1}}{dx} \tag{3.16}$$

$$+ M_7 \frac{d^3P^j}{dx^3} + M_8 \frac{d^3P^{j-1}}{dx^3} + M_9(Q^j - Q^{j-1}) + M_{10}(\frac{d^2Q^j}{dx^2} - \frac{d^2Q^{j-1}}{dx^2}) = 0$$

where

$$M_1 = G_2g_1 - G_1g_2 \; ; \qquad M_2 = G_2g_3 \; ; \tag{3.17}$$

$$M_3 = G_2g_4 - G_3g_2 \; ; \qquad M_4 = - G_4g_2 \; ;$$

$$M_5 = G_2g_5 \; ; \qquad M_6 = G_2g_6 \; ;$$

$$M_7 = G_2g_7 - G_5g_2 \; ; \qquad M_8 = G_2g_8 - G_6g_2 \; ;$$

$$M_9 = G_2g_9 \; ; \qquad M_{10} = - G_7g_2 \; .$$

When equation (3.16) is differentiated and the resulting $\frac{d^2U}{dx^2}$ is substituted into equation (3.14), one obtains

$$U = m_1 \frac{dW}{dx} + m_2 \frac{d^3W}{dx^3} + m_3 \frac{d^5W}{dx^5} + m_4 \frac{d^2P^j}{dx^2} + m_5 \frac{d^2P^{j-1}}{dx^2} \tag{3.18}$$

$$+ m_6 \frac{d^4P^j}{dx^4} + m_7 \frac{d^4P^{j-1}}{dx^4} + m_8(\frac{dQ^j}{dx} - \frac{dQ^{j-1}}{dx}) + m_9(\frac{d^3Q^j}{dx^3} - \frac{d^3Q^{j-1}}{dx^3})$$

where

$$m_1 = (M_2G_2 - M_1G_3)/G_1M_1 \ , \tag{3.19}$$

$$m_2 = (M_3G_2 - M_1G_4)/G_1M_1 \ ,$$

$$m_3 = M_4G_2/G_1M_1 \ ,$$

$$m_4 = (M_5G_2 - M_1G_5)/G_1M_1 \ ,$$

$$m_5 = (M_6G_2 - M_1G_6)/G_1M_1 \ ,$$

$$m_6 = M_7G_2/G_1M_1 \ ,$$

$$m_7 = M_8G_2/G_1M_1 \ ,$$

$$m_8 = (M_9G_2 - M_1G_7)/G_1M_1 \ ,$$

$$m_9 = M_{10}G_2/G_1M_1 \ .$$

Differentiating equation (3.18) and substituting the results into equation (3.16), one obtains an equation which only involves the normal displacement coordinate function W and the surface loads, which is

$$\frac{d^6W}{dx^6} + 3s_2\frac{d^4W}{dx^4} + 3s_1\frac{d^2W}{dx^2} + s_0W = s_3\frac{dP^{j-1}}{dx} + s_4\frac{d^3P^{j-1}}{dx^3} \tag{3.20}$$

$$+ s_5\frac{d^5P^{j-1}}{dx^5} + s_6\frac{dP^j}{dx} + s_7\frac{d^3P^j}{dx^3} + s_8\frac{d^5P^j}{dx^5} + s_9(Q^j - Q^{j-1})$$

$$+ s_{10}\left(\frac{d^2Q^j}{dx^2} - \frac{d^2Q^{j-1}}{dx^2}\right) + s_{11}\left(\frac{d^4Q^j}{dx^4} - \frac{d^4Q^{j-1}}{dx^4}\right)$$

where

$$s_0 = M_2/m_0 \ ; \qquad s_1 = (M_3 + m_1M_1)/3m_0 \ ; \quad (3.21)$$

$$s_2 = (M_4 + M_1m_2)/3m_0 \ ; \quad s_3 = -M_6/m_0 \ ;$$

$$s_4 = -(M_8 + M_1m_5)/m_0 \ ; \quad s_5 = -M_1m_7/m_0 \ ;$$

$$s_6 = -M_5/m_0 \ ; \qquad s_7 = -(M_7 + M_1m_4)/m_0 \ ;$$

$$s_8 = -M_1m_6/m_0 \ ; \qquad s_9 = -M_9/m_0 \ ;$$

$$s_{10} = -(M_{10} + M_1m_8)/m_0 \ ,$$

and where

$$m_0 = M_1m_3 \ .$$

Since the magnitudes of the $C_4^j$ and $F_4^j$ in equations (3.1) and (3.3) are of the order of $h^5$, one may neglect these two terms in comparison to the other coefficients (order of $h^3$ or lower) for thin shell analysis. By doing so, and by using the same procedures presented previously, equation (3.9) becomes

$$\Phi = f_2 \frac{d^2U}{dx^2} + f_3 \frac{d^3W}{dx^3} + f_4P^j + f_5P^{j-1} + f_6 \frac{d^2P^j}{dx^2} + f_7 \frac{d^2P^{j-1}}{dx^2} \quad (3.9a)$$

where

$$f_2 = -F_1/F_3 \ ; \qquad f_3 = -F_2/F_3 \ ; \qquad (3.10a)$$

$$f_4 = -F_5/F_3 \ ; \qquad f_5 = -F_6/F_3 \ ;$$

$$f_6 = -F_7/F_3 \ ; \qquad f_7 = -F_8/F_3 \ .$$

The constants $g_i$ and $G_i$ in equations (3.13) and (3.15) have new expressions and they are

$$g_1 = D_1 ; \qquad g_2 = D_3 f_2 ; \qquad (3.13a)$$

$$g_3 = D_2 - \bar{\lambda}^2 B_{11} h ; \qquad g_4 = D_3 f_3 ;$$

$$g_5 = D_4 + D_3 f_4 ; \qquad g_6 = D_5 + D_3 f_5 ;$$

$$g_7 = D_3 f_6 ; \qquad g_8 = D_3 f_7 ;$$

$$g_9 = 1 ,$$

and

$$G_1 = \bar{\lambda}^2 B_{11} h ; \qquad G_2 = C_1 ;$$

$$G_3 = C_2 ; \qquad G_4 = C_3 ;$$

$$G_5 = C_5 ; \qquad G_6 = C_6 ;$$

$$G_7 = 0 .$$

Equations (3.18) and (3.20) then reduce to the following expressions

$$U = m_1 \frac{dW}{dx} + m_2 \frac{d^3W}{dx^3} + m_3 \frac{d^5W}{dx^5} + m_4 \frac{d^2P^j}{dx^2} + m_5 \frac{d^2P^{j-1}}{dx^2} \qquad (3.18a)$$

$$+ m_6 \frac{d^4P^j}{dx^4} + m_7 \frac{d^4P^{j-1}}{dx^4} + m_8 \left(\frac{dQ^j}{dx} - \frac{dQ^{j-1}}{dx}\right) ,$$

$$\frac{d^6W}{dx^6} + 3s_2 \frac{d^4W}{dx^4} + 3s_1 \frac{d^2W}{dx^2} + s_0W = s_3 \frac{dP^{j-1}}{dx} + s_4 \frac{d^3P^{j-1}}{dx^3} \tag{3.20a}$$

$$+ s_5 \frac{d^5P^{j-1}}{dx^5} + s_6 \frac{dP^j}{dx} + s_7 \frac{d^3P^j}{dx^3} + s_8 \frac{d^5P^j}{dx^5} + s_9(Q^j - Q^{j-1})$$

$$+ s_{10}(\frac{d^2Q^j}{dx^2} - \frac{d^2Q^{j-1}}{dx^2})$$

where $M_i$, $m_i$, and $s_i$ are indicated in equations (3.17), (3.19) and (3.21).

## Rotationally Symmetric Vibration of Single-layered Shell

For the free rotationally symmetric vibration of an orthotropic cylindrical single-layered shell, one may set the surface loading terms $X^0$, $X^1$, $Z^0$ and $Z^1$ equal to zero. Therefore, equations (3.20), (3.18) and (3.9) become

$$\frac{d^6W}{dx^6} + 3s_2 \frac{d^4W}{dx^4} + 3s_1 \frac{d^2W}{dx^2} + s_0W = 0 , \tag{3.22}$$

$$U = m_1 \frac{dW}{dx} + m_2 \frac{d^3W}{dx^3} + m_3 \frac{d^5W}{dx^5} \tag{3.23}$$

and

$$\Phi = f_1U + f_2 \frac{d^2U}{dx^2} + f_3 \frac{dW}{dx} . \tag{3.24}$$

It is clear that once the solution for W is found, the solutions for U and $\Phi$ can be obtained immediately. In order to find the solution for the normal displacement coordinate function W, let $W = e^{\lambda x}$ and substitute into equation (3.22). This procedure yields the following characteristic equation:

$$\lambda^6 + 3s_2\lambda^4 + 3s_1\lambda^2 + s_0 = 0 \quad . \tag{3.25}$$

The substitution $\lambda^2 = \xi$ reduces (3.25) to the cubic equation

$$\xi^3 + 3s_2\xi + 3s_1\xi + s_0 = 0 \quad . \tag{3.26}$$

If one defines

$$q_1 = s_1 - s_2^2, \tag{3.27}$$

$$q_2 = \frac{1}{2}(3s_1 s_2 - s_0) - s_2^3,$$

$$q_3 = [q_2 + (q_1^3 + q_2^2)^{1/2}]^{1/3} ,$$

$$q_4 = [q_2 - (q_1^3 + q_2^2)^{1/2}]^{1/3} ,$$

the roots of equation (3.26) can be written as [15]

$$\xi_1 = q_3 + q_4 - s_2 , \tag{3.28}$$

$$\xi_2 = -\frac{1}{2}(q_3 + q_4) - s_2 + \frac{i\sqrt{3}}{2}(q_3 - q_4),$$

$$\xi_3 = -\frac{1}{2}(q_3 + q_4) - s_2 - \frac{i\sqrt{3}}{2}(q_3 - q_4)$$

where $i = \sqrt{-1}$.

The solutions for W, U and $\Phi$ will have different expressions depending on whether the roots $\lambda_k$ of the characteristic equation (3.25) are real, pure imaginary, complex conjugates or multiple roots. The general solution of equation (3.25), together with the different expressions of the solution of W, U and Φ are presented and discussed in detail in Appendix B. Here, for the convenience of further analysis, one may express the solutions of W, U and Φ in general forms as follows:

$$W = \sum_{\mu=1}^{6} K_{1\mu} e^{\lambda_\mu x} , \tag{3.29}$$

$$U = \sum_{\mu=1}^{6} K_{2\mu} e^{\lambda_\mu x}$$

and

$$\Phi = \sum_{\mu=1}^{6} K_{3\mu} e^{\lambda_\mu x}$$

where the $K_{i\mu}$ are constants and the $\lambda_\mu$ are roots of equation (3.25), considered complex in general.

## Boundary Conditions

Associated with the system of equations (3.6), (3.7) and (3.8) are six boundary conditions to be satisfied at the edges of the cylindrical shell. These conditions are identical to those of the usual linear shell theory. However, it should be noted that in the case of orthotropy, the expressions of the boundary forces in terms of displacement components and the shear stress function are quite different when compared with isotropic shells.

Simply Supported Edge. For rotationally symmetric vibration, the boundary conditions corresponding to the simply supported condition are

$$u_o = 0 \,, \tag{3.30}$$

$$w = 0 \,,$$

$$M_x = 0 \,.$$

The condition $M_x = 0$ at the supported edge is equivalent to the condition

$$\frac{\partial u_o}{\partial x} - R\,\frac{\partial^2 w}{\partial x^2} + \frac{a_{55}Rh^2}{5}\,\frac{\partial \varphi}{\partial x} = 0 \quad .$$

Therefore, the boundary conditions for the simply supported condition may be written as

$$U = 0 \,, \tag{3.31}$$

$$W = 0 \,,$$

$$\frac{dU}{dx} - R\,\frac{d^2W}{dx^2} + \frac{a_{55}Rh^2}{5}\,\frac{d\Phi}{dx} = 0 \,.$$

Fixed Edge. The boundary conditions corresponding to a fixed edge are

$$u_o = 0 \,, \tag{3.32}$$

$$w = 0 \,,$$

$$\frac{\partial w}{\partial x} = 0 \,;$$

or, equivalently,

$$U = 0 \ , \tag{3.32a}$$

$$W = 0 \ ,$$

$$\frac{dW}{dx} = 0 \ .$$

Free Edge Support

The boundary conditions corresponding to a free edge support are

$$T_x = 0 \ , \tag{3.33}$$

$$N_x = 0 \ ,$$

$$M_x = 0 \ ;$$

or, equivalently,

$$C_1 \frac{dU}{dx} + C_2 W + C_3 \frac{d^2W}{dx^2} + C_4 \frac{d\Phi}{dx} + C_5 \frac{dP^+}{dx} + C_6 \frac{dP^-}{dx} = 0, \tag{3.33a}$$

$$F_3 \Phi + F_5 P^+ + F_6 P^- = 0 \ ,$$

$$\frac{dU}{dx} - R \frac{d^2W}{dx^2} + \frac{a_{55} R h^2}{5} \frac{d\Phi}{dx} = 0 \ .$$

The above alternatives of the boundary conditions do not exhaust all possibilities. Boundary conditions may assume a great variety of forms depending on the character of the supports on which the shell rests.

Frequency Equation of a Single-Layered Shell

For a cylindrical shell having one single layer only, the frequency equation can be obtained immediately by using the general results discussed in this chapter. In order to obtain the frequency equation of a single-layered shell, one substitutes the solutions for W, U, and $\Phi$ into the proper boundary conditions, given in the previous section, and requires that the determinant of the coefficients for the six resulting simultaneous equations will vanish. These equations may be written in the following general form:

$$[\bar{D}]\begin{Bmatrix} K_1 \\ K_2 \\ K_3 \\ K_4 \\ K_5 \\ K_6 \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} \\ D_{21} & \cdot & \cdot & \cdot & \cdot & D_{26} \\ \vdots & & & & & \vdots \\ \vdots & & & & & \vdots \\ D_{61} & \cdot & \cdot & \cdot & \cdot & D_{66} \end{bmatrix} \begin{Bmatrix} K_1 \\ K_2 \\ K_3 \\ K_4 \\ K_5 \\ K_6 \end{Bmatrix} = \{0\}. \quad (3.33)$$

As indicated earlier, the forms of the solutions of W, U and $\Phi$ depend on the roots of characteristic equation (3.25), which in turn depend on the geometry and material properties of the shell. The expressions for the elements of the coefficient matrix are therefore different for different combinations of geometrical and material parameters, and boundary conditions.

When the supporting conditions along both edges of a shell are identically the same, the vibration modes will be either symmetrical or antisymmetrical about the mid-section of the shell. If the origin is placed in this plane of symmetry, then equation (3.33) will be degenerated into two sets of three simultaneous homogeneous algebraic equations. The vanishing of the determinant of the coefficient matrix for one set of equations will yield the frequency equation corresponding to symmetrical modes of vibration while the other set of equations will give the frequency equation corresponding to antisymmetrical modes. The two sets of equations are

$$[\overline{D}_1] \begin{Bmatrix} K_1 \\ K_2 \\ K_3 \end{Bmatrix} = \{0\} \quad \text{and} \quad [\overline{D}_2] \begin{Bmatrix} K_4 \\ K_5 \\ K_6 \end{Bmatrix} = \{0\} \tag{3.34}$$

where

$$[\overline{D}_1] = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix} ; \quad [\overline{D}_2] = \begin{bmatrix} D_{14} & D_{15} & D_{16} \\ D_{24} & D_{25} & D_{26} \\ D_{34} & D_{35} & D_{36} \end{bmatrix} . \tag{3.35}$$

In this chapter, a sixth-order linear ordinary differential equation governing the rotationally symmetric motion of a general orthotropic layer has been derived. The boundary conditions and the general frequency equation of a shell having only one layer subjected to any homogeneous boundary conditions were presented. For specific fixed-fixed supporting edges, the frequency equation corresponding to all possible forms of solutions of the characteristic equation (3.25) was obtained.

# CHAPTER IV

## ROTATIONALLY SYMMETRIC VIBRATIONS OF CYLINDRICAL SHELLS HAVING N ORTHOTROPIC LAMINAE

The present analysis considers each layer of a layered shell individually. The interaction between layers is coupled through the compatibility conditions for deformation and stresses. The analysis allows the estimate of bond stresses between layers which is important practically. To determine the natural frequency of an N-layered orthotropic cylindrical shell, N sets of differential equations, together with N sets of boundary conditions along the edges and N-1 sets of continuity conditions between every two adjacent layers, must be satisfied simultaneously.

### General Solutions of a jth Layer of a Layered Shell

As shown in the first section of the previous chapter, the set of differential equations of a jth layer of a layered shell are

$$\frac{d^6W^j}{dx^6} + 3s_2^j\frac{d^4W^j}{dx^4} + 3s_1^j\frac{d^2W^j}{dx^2} + s_0^j\frac{dW^j}{dx} = s_3^j\frac{dP^{j-1}}{dx} \tag{4.1}$$

$$+ s_4^j\frac{d^3P^{j-1}}{dx^3} + s_5^j\frac{d^5P^{j-1}}{dx^5} + s_6^j\frac{dP^j}{dx} + s_7^j\frac{d^3P^j}{dx^3}$$

$$+ s_8^j\frac{d^5P^j}{dx^5} + s_9^j(Q^j - Q^{j-1}) + s_{10}^j\left(\frac{d^2Q^j}{dx^2} - \frac{d^2Q^{j-1}}{dx^2}\right)$$

$$+ s_{11}^j\left(\frac{d^4Q^j}{dx^4} - \frac{d^4Q^{j-1}}{dx^4}\right),$$

$$U^j = m_1^j \frac{dW^j}{dx} + m_2^j \frac{d^3W^j}{dx^3} + m_3^j \frac{d^5W^j}{dx^5} + m_4^j \frac{d^2P^j}{dx^2} + m_5^j \frac{d^2P^{j-1}}{dx^2} \quad (4.2)$$

$$+ m_6^j \frac{d^4P^j}{dx^4} + m_7^j \frac{d^4P^{j-1}}{dx^4} + m_8^j \left(\frac{dQ^j}{dx} - \frac{dQ^{j-1}}{dx}\right)$$

$$+ m_9^j \left(\frac{d^3Q^j}{dx^3} - \frac{d^3Q^{j-1}}{dx^3}\right),$$

$$\Phi^j = f_1^j U^j + f_2^j \frac{d^2U^j}{dx^2} + f_3^j \frac{dW^j}{dx} + f_4^j P^j + f_5^j P^{j-1} + f_6^j \frac{d^2P^j}{dx^2} \quad (4.3)$$

$$+ f_7^j \frac{d^2P^{j-1}}{dx^2} .$$

Assume that the loading functions $P^j(x)$ and $Q^j(x)$ at the jth interface may be represented in the forms

$$P^j(x) = \sum_{n=1}^{\infty} b_n^j \sin \frac{n\pi x}{\ell} \quad (4.4)$$

and

$$Q^j(x) = \sum_{n=0}^{\infty} d_n^j \cos \frac{n\pi x}{\ell} . \quad (4.5)$$

The substitution of equations (4.4) and (4.5) into equation (4.1) results in the equation

$$\frac{d^6W^j}{dx^6} + 3s_2^j \frac{d^4W^j}{dx^4} + 3s_1^j \frac{d^2W^j}{dx^2} + s_0^j W^j = \sum_{n=0}^{\infty} (A_{1n}^j b_n^j + \quad (4.6)$$

$$+ A_{2n}^j b_n^{j-1} + A_{3n}^j d_n^j + A_{4n}^j d_n^{j-1}) \cos \frac{n\pi x}{\ell}$$

where $b_0^j = b_0^{j-1} = 0$, and where

$$A_{1n}^j = s_6^j\left(\frac{n\pi}{\ell}\right) - s_7^j\left(\frac{n\pi}{\ell}\right)^3 + s_8^j\left(\frac{n\pi}{\ell}\right)^5 , \tag{4.7}$$

$$A_{2n}^j = s_3^j\left(\frac{n\pi}{\ell}\right) - s_4^j\left(\frac{n\pi}{\ell}\right)^3 + s_5^j\left(\frac{n\pi}{\ell}\right)^5 ,$$

$$A_{3n}^j = s_9^j - s_{10}^j\left(\frac{n\pi}{\ell}\right)^2 + s_{11}^j\left(\frac{n\pi}{\ell}\right)^4 ,$$

$$A_{4n}^j = - A_{3n}^j .$$

The general forms of the homogeneous solution of equation (4.6) are represented by equation (3.29). The exact expression of the general solution depends on the roots of characteristic equation (3.25) as to whether they are real, pure imaginary,complex conjugates or multiple roots. For convenience of discussion, the homogeneous solution in general form is rewritten as follows:

$$W_H^j = \sum_{\mu=1}^{6} K_\mu^j e^{\lambda_\mu^j x} \tag{4.8}$$

where $\lambda_\mu^j$, $\mu = 1,2,3,\ldots,6$, are the roots of equation (3.25) which are complex numbers in general.

The particular solution of equation (4.6) is

$$W_P^j = \sum_{n=0}^{\infty} T_n^j (A_{1n}^j b_n^j + A_{2n}^j b_n^{j-1} + A_{3n}^j d_n^j + A_{4n}^j d_n^{j-1}) \cos \frac{n\pi x}{\ell} \tag{4.9}$$

where

$$T_n^j = -\left\{\left(\frac{n\pi}{\ell}\right)^6 - 3s_2^j\left(\frac{n\pi}{\ell}\right)^4 + 3s_1^j\left(\frac{n\pi}{\ell}\right)^2 - s_0^j\right\}^{-1} . \qquad (4.10)$$

Therefore, the general solution of equation (4.6) may be written as

$$W^j = \sum_{\mu=1}^{6} K_\mu^j e^{\lambda_\mu^j x} + \sum_{n=0}^{\infty} T_n^j (A_{1n}^j b_n^j + A_{2n}^j b_n^{j-1} + A_{3n}^j d_n^j$$
$$+ A_{4n}^j d_n^{j-1}) \cos \frac{n\pi x}{\ell} . \qquad (4.11)$$

Upon substitution of equation (4.11) into equations (4.2) and (4.3), one obtains the following general solutions for the longitudinal displacement coordinate function U and transverse shear function:

$$U^j = \sum_{\mu=1}^{6} \bar{c}_\mu^j K_\mu^j e^{\lambda_\mu^j x} + \sum_{n=1}^{\infty} (\bar{a}_{1n}^j b_n^j + \bar{a}_{2n}^j b_n^{j-1} + \bar{a}_{3n}^j d_n^j$$
$$+ \bar{a}_{4n}^j d_n^{j-1}) \sin \frac{n\pi x}{\ell} \qquad (4.12)$$

and

$$\Phi^j = \sum_{\mu=1}^{6} \bar{f}_\mu^j K_\mu^j e^{\lambda_\mu^j x} + \sum_{n=1}^{\infty} (\bar{g}_{1n}^j b_n^j + \bar{g}_{2n}^j b_n^{j-1} + \bar{g}_{3n}^j d_n^j$$
$$+ \bar{g}_{4n}^j d_n^{j-1}) \sin \frac{n\pi x}{\ell} \qquad (4.13)$$

where

$$\bar{c}_\mu^j = m_1^j \lambda_\mu^j + m_2^j (\lambda_\mu^j)^3 + m_3^j (\lambda_\mu^j)^5 , \quad \text{for } \mu = 1,2,3,\ldots,6 , \qquad (4.14)$$

$$\bar{a}_{1n}^j = -T_n^j A_{1n}^j \left\{ m_1^j\left(\frac{n\pi}{\ell}\right) - m_2^j\left(\frac{n\pi}{\ell}\right)^3 + m_3^j\left(\frac{n\pi}{\ell}\right)^5 \right\} - m_4^j\left(\frac{n\pi}{\ell}\right)^2 + m_6^j\left(\frac{n\pi}{\ell}\right)^4 ,$$

$$\bar{a}^j_{2n} = -\ T^j_n A^j_{2n} \left\{ m^j_1(\frac{n\pi}{\ell}) - m^j_2(\frac{n\pi}{\ell})^3 + m^j_3(\frac{n\pi}{\ell})^5 \right\} - m^j_5(\frac{n\pi}{\ell})^2 + m^j_7(\frac{n\pi}{\ell})^4 ,$$

$$\bar{a}^j_{3n} = -\ T^j_n A^j_{3n} \left\{ m^j_1(\frac{n\pi}{\ell}) - m^j_2(\frac{n\pi}{\ell})^3 + m^j_3(\frac{n\pi}{\ell})^5 \right\} - m^j_8(\frac{n\pi}{\ell}) + m^j_9(\frac{n\pi}{\ell})^3 ,$$

$$\bar{a}^j_{4n} = -\ \bar{a}^j_{3n}$$

$$\bar{f}^j_\mu = f^j_1\ \bar{c}^j_\mu + f^j_2\ \bar{c}^j_\mu\ (\lambda^j_\mu)^2 + f^j_3\lambda^j_\mu \quad \text{for} \quad \mu = 1,2,3,\ldots,6 ,$$

$$\bar{g}^j_{1n} = \left\{ f^j_1 - f^j_2(\frac{n\pi}{\ell})^2 \right\} \bar{a}^j_{1n} - f^j_3\ T^j_n A^j_{1n}(\frac{n\pi}{\ell}) + f^j_4 - f^j_6(\frac{n\pi}{\ell})^2 ,$$

$$\bar{g}^j_{2n} = \left\{ f^j_1 - f^j_2(\frac{n\pi}{\ell})^2 \right\} \bar{a}^j_{2n} - f^j_3\ T^j_n A^j_{2n}(\frac{n\pi}{\ell}) + f^j_5 - f^j_7(\frac{n\pi}{\ell})^2 ,$$

$$\bar{g}^j_{3n} = \left\{ f^j_1 - f^j_2(\frac{n\pi}{\ell})^2 \right\} \bar{a}^j_{3n} - f^j_3\ T^j_n A^j_{3n}(\frac{n\pi}{\ell}) ,$$

$$\bar{g}^j_{4n} = -\ \bar{g}^j_{3n} .$$

If $C^j_4$ and $E^j_4$ are neglected as discussed in page 29 of Chapter III, then equation (3.9a) may be used instead of equation (3.9), and the coefficients $\bar{f}^j_\mu$ and $\bar{g}^j_{\mu n}$ then are

$$\bar{f}^j_\mu = f^j_2\ \bar{c}^j_\mu(\lambda^j_\mu)^2 + f^j_3(\lambda^j_\mu)^3 \quad \text{for} \quad \mu = 1,2,3,\ldots,6 , \tag{4.14a}$$

$$\bar{g}^j_{1n} = -f^j_2\ \bar{a}^j_{2n}(\frac{n\pi}{\ell})^2 + f^j_3 T^j_n A^j_{1n}(\frac{n\pi}{\ell})^3 + f^j_4 - f^j_6(\frac{n\pi}{\ell})^2 ,$$

$$\bar{g}^j_{2n} = -f^j_2\ \bar{a}^j_{2n}(\frac{n\pi}{\ell})^2 + f^j_3 T^j_n A^j_{2n}(\frac{n\pi}{\ell})^3 + f^j_5 - f^j_7(\frac{n\pi}{\ell})^2 ,$$

$$\bar{g}^j_{3n} = -f^j_2\ \bar{a}^j_{3n}(\frac{n\pi}{\ell})^2 + f^j_3 T^j_n A^j_{3n}(\frac{n\pi}{\ell})^3 ,$$

$$\bar{g}^j_{4n} = -\bar{g}^j_{3n} .$$

Upon substitution of equations (4.11), (4.12) and (4.13) into the proper boundary conditions presented in Chapter III, one obtains the following general expressions:

$$[\overline{D}^j]\begin{Bmatrix} K_1^j \\ K_2^j \\ K_3^j \\ K_4^j \\ K_5^j \\ K_6^j \end{Bmatrix} = \begin{Bmatrix} p_1^j \\ p_2^j \\ p_3^j \\ p_4^j \\ p_5^j \\ p_6^j \end{Bmatrix} = \{\overline{p}^j\} . \tag{4.15}$$

The elements of the coefficient matrix $[\overline{D}^j]$ will have different expressions depending on the boundary conditions along the edges as well as the roots of the characteristic equation. The matrix $\{\overline{p}^j\}$ will depend on the boundary conditions only. For example, if a shell is fixed at both ends, the matrix $\{\overline{p}^j\}$ becomes

$$\{\overline{p}^j\} = \begin{Bmatrix} -\sum_{n=0}^{\infty} T_n^j(A_{1n}^j b_n^j + A_{2n}^j b_n^{j-1} + A_{3n}^j d_n^j + A_{4n}^j d_n^{j-1}) \\ 0 \\ 0 \\ -\sum_{n=0}^{\infty} (-1)^n T_n^j(A_{1n}^j b_n^j + A_{2n}^j b_n^{j-1} + A_{3n}^j d_n^j + A_{4n}^j d_n^{j-1}) \\ 0 \\ 0 \end{Bmatrix} . \tag{4.16}$$

Since the matrix $[\overline{D}^j]$ is nonsingular,* one may express the unknown constants of integration $K_\mu^j$, $\mu = 1,2,3,\ldots,6$, in terms of the unknown Fourier coefficients $b_n^j$, $b_n^{j-1}$, $d_n^j$ and $d_n^{j-1}$, $n = 1,2,3,\ldots,\infty$, by inverting the matrix $[\overline{D}^j]$, i.e.

$$\{\overline{K}^j\} = [\overline{D}^j]^{-1}\{\overline{p}^j\} \tag{4.17}$$

or

$$\{\overline{K}^j\} = [\overline{\xi}^j]\{\overline{p}^j\} . \tag{4.17a}$$

In equation (4.8), the $e^{\lambda_\mu^j x}$, $\mu = 1,2,3,\ldots,6$, are continuous functions. Hence they can be represented by their Fourier sine or cosine expansions, and they are

$$e^{\lambda_\mu^j x} = \sum_{n=0}^{\infty} t_{\mu n}^j \cos\frac{n\pi x}{\ell}\ , \quad \text{for } \mu = 1,2,3,\ldots,6, \tag{4.18}$$

or

$$e^{\lambda_\mu^j x} = \sum_{n=1}^{\infty} r_{\mu n}^j \sin\frac{n\pi x}{\ell}\ , \quad \text{for } \mu = 1,2,3,\ldots,6, \tag{4.18a}$$

where

$$t_{\mu 0}^j = \frac{1}{\ell}\int_0^\ell e^{\lambda_\mu^j x}\,dx\ , \quad \text{for } \mu = 1,2,3,\ldots,6\ , \tag{4.19a}$$

---

*If $[\overline{D}^j]$ is a singular matrix, the implication is that either the stiffness of the shell is zero or the shell has infinite displacement.

$$t^j_{\mu n} = \frac{2}{\ell}\int_0^{\ell} e^{\lambda^j_\mu x} \cos\frac{n\pi x}{\ell}\, dx \ , \quad \text{for } \mu = 1,2,3,\ldots,6 \ , \quad n = 1,2,3,\ldots,\infty \ . \tag{4.19b}$$

and

$$r^j_{\mu n} = \frac{2}{\ell}\int_0^{\ell} e^{\lambda^j_\mu x} \sin\frac{n\pi x}{\ell}\, dx \ , \quad \text{for } \mu = 1,2,3,\ldots,6 \ , \quad n = 1,2,3,\ldots,\infty \ . \tag{4.19c}$$

The results of the substitution of equation (4.18), in conjunction with equation (4.19), into equations (4.11), (4.12) and (4.13), are

$$W^j = \sum_{n=0}^{\infty}\left\{ \sum_{\mu=1}^{6} K^j_\mu\, t^j_{n\mu} + T^j_n(A^j_{1n}b^j_n + A^j_{2n}b^{j-1}_n + A^j_{3n}d^j_n + A^j_{4n}d^{j-1}_n) \right\}\cos\frac{n\pi x}{\ell} \ , \tag{4.20}$$

$$U^j = \sum_{n=1}^{\infty}\left\{ \sum_{\mu=1}^{6} \bar{c}^j_\mu\, r^j_{\mu n}K^j_\mu + \bar{a}^j_{1n}b^j_n + \bar{a}^j_{2n}b^{j-1}_n + \bar{a}^j_{3n}d^j_n + \bar{a}^j_{4n}d^{j-1}_n \right\}\sin\frac{n\pi x}{\ell} \ , \tag{4.21}$$

and

$$\Phi^j = \sum_{n=1}^{\infty}\left\{ \sum_{\mu=1}^{6} \bar{f}^j_\mu r^j_{\mu n}K^j_n + \bar{g}^j_{1n}b^j_n + \bar{g}^j_{2n}b^{j-1}_n + \bar{g}^j_{3n}d^j_n + \bar{g}^j_{4n}\, d^{j-1}_n \right\}\sin\frac{n\pi x}{\ell} \ , \tag{4.22}$$

where the coefficients $\bar{c}^j_\mu$ and $\bar{f}^j_\mu$, $\mu = 1,2,3,\ldots,6$, depend on the boundary conditions as well as the roots of equation (3.25).

The substitution of equation (4.17a) into equations (4.20), (4.21) and (4.22) results in the following simple expressions for $W^j$, $U^j$ and $\Phi^j$:

$$W^j = \sum_{n=0}^{\infty} \sum_{\nu=0}^{\infty} \left\{ \zeta^j_{1n\nu} b^j_\nu + \zeta^j_{2n\nu} b^{j-1}_\nu + \zeta^j_{3n\nu} d^j_\nu + \zeta^j_{4n\nu} d^{j-1}_\nu \right\} \cos \frac{n\pi x}{\ell}, \quad (4.23)$$

$$U^j = \sum_{n=1}^{\infty} \sum_{\nu=0}^{\infty} \left\{ \bar{\alpha}^j_{1n\nu} b^j_\nu + \bar{\alpha}^j_{2n\nu} b^{j-1}_\nu + \bar{\alpha}^j_{3n\nu} d^j_\nu + \bar{\alpha}^j_{4n\nu} d^{j-1}_\nu \right\} \sin \frac{n\pi x}{\ell} \quad (4.24)$$

and

$$\Phi^j = \sum_{n=1}^{\infty} \sum_{\nu=0}^{\infty} \left\{ J^j_{1n\nu} b^j_\nu + J^j_{2n\nu} b^{j-1}_\nu + J^j_{3n\nu} d^j_\nu + J^j_{4n\nu} d^{j-1}_\nu \right\} \sin \frac{n\pi x}{\ell} \quad (4.25)$$

where $\zeta^j_{\mu n\nu}$, $\bar{\alpha}^j_{\mu n\nu}$ and $J^j_{\mu n\nu}$, $\mu = 1,2,3,4$, are constants depending on the roots of characteristic equation (3.25) and the boundary conditions.

## Continuity Conditions and Frequency Equation

From the assumption that there is no slippage between two adjacent layers, the following continuity conditions (or contact conditions) at all interfaces must be satisfied:

$$w^j\Big|_{z=\frac{h^j}{2}} = w^{j+1}\Big|_{z=-\frac{h^{j+1}}{2}}, \quad \text{for } j = 1,2,3,\ldots,N-1\,, \quad (4.26)$$

and

$$u^j\Big|_{z=\frac{h^j}{2}} = u^{j+1}\Big|_{z=-\frac{h^{j+1}}{2}}, \quad \text{for } j = 1,2,3,\ldots,N-1\,. \quad (4.27)$$

Since the normal displacement $w^j$ is independent of the normal coordinate z, equation (4.26) is equivalent to the expressions

$$W^j(x) = W^{j-1}(x) \ , \qquad \text{for } j = 1,2,3,\ldots,N-1 \ . \tag{4.28}$$

From equations (2.8) and (3.5), the second set of contact conditions (4.27) becomes

$$U^j - \frac{h^j}{2}\frac{dW^j}{dx} + \frac{a_{55}^j (h^j)^3}{12}\Phi^j + \frac{3}{8} a_{55}^j h^j p^j - \frac{1}{8} a_{55}^j h^j p^{j-1} \tag{4.29}$$

$$= U^{j+1} + \frac{h^{j+1}}{2}\frac{dW^{j+1}}{dx} - \frac{a_{55}^{j+1}(h^{j+1})^3}{12}\Phi^{j+1} - \frac{1}{8} a_{55}^{j+1} h^{j+1} p^{j+1}$$

$$+ \frac{3}{8} a_{55}^{j+1} h^{j+1} p^j \ .$$

The substitution of equation (4.23) into equation (4.28) results in the equation

$$\sum_{n=0}^{\infty}\sum_{v=0}^{\infty}\Big\{\zeta_{2nv}^j b_v^{j-1} + (\zeta_{1nv}^j - \zeta_{2nv}^{j+1}) b_v^j - \zeta_{1nv}^{j+1} b_v^{j+1} \tag{4.30}$$

$$+ \zeta_{4nv}^j d_v^{j-1} + (\zeta_{3nv}^j - \zeta_{4nv}^{j+1}) d_v^j - \zeta_{3nv}^{j+1} d_v^{v+1}\Big\}\cos\frac{n\pi x}{\ell} = 0$$

$$\text{for } j = 1,2,3,\ldots,N-1 \ .$$

Since this set of equations must be satisfied for any arbitrary value of x, each coefficient of the infinite series must be identically equal to zero; this yields the following set of infinitely many simultaneous algebraic equations:

$$\sum_{v=0}^{\infty}\Big\{L_{1nv}^j b_v^{j-1} + L_{2nv}^j b_v^j + L_{3nv}^j b_v^{j+1} + L_{4nv}^j d_v^{j-1} + L_{5nv}^j d_v^j + L_{6nv}^j d_v^{j+1}\Big\} = 0 \tag{4.31}$$

$$\text{for } n = 0,1,2,\ldots,\infty \ ,$$

$$j = 1,2,3,\ldots,N-1 \ .$$

where

$$L^j_{1n} = \zeta^j_{2n\nu} , \tag{4.32}$$

$$L^j_{2n} = \zeta^j_{1n\nu} - \zeta^{j+1}_{2n\nu} ,$$

$$L^j_{3n} = - \zeta^{j+1}_{1n\nu} ,$$

$$L^j_{4n} = \zeta^j_{4n\nu} ,$$

$$L^j_{5n} = \zeta^j_{3n\nu} - \zeta^{j+1}_{4n\nu} ,$$

$$L^j_{6n} = - \zeta^{j+1}_{3n\nu} .$$

The substitution of equations (4.23), (4.24), and (4.25), together with equations (4.4) and (4.5), into equation (4.29) yields the equation

$$\sum_{n=1}^{\infty} \sum_{\nu=0}^{\infty} \left\{ I^j_{1n\nu} b^{j-1}_\nu + I^j_{2n\nu} b^j_\nu + I^j_{3n\nu} b^{j+1}_\nu + I^j_{4n\nu} d^{j-1}_\nu + I^j_{5n\nu} d^j_\nu \right. \tag{4.33}$$

$$\left. + I^j_{6n\nu} d^{j+1}_\nu \right\} \sin \frac{n\pi x}{\ell} = 0$$

for $j = 1,2,3,\ldots,N-1$ ,

where

$$I^j_{1n\nu} = -\left\{ \bar{\alpha}^j_{2n\nu} + \frac{h^j}{2} \left(\frac{n\pi}{\ell}\right)\zeta^j_{2n\nu} + \frac{a^j_{55}(h^j)^3}{12} J^j_{2n\nu} - \frac{1}{8} a^j_{55} h^j \delta_{n\nu} \right\}, \tag{4.34}$$

$$I^j_{2n\nu} = \bar{\alpha}^{j+1}_{2n\nu} - \frac{h^{j+1}}{2} \left(\frac{n\pi}{\ell}\right)\zeta^{j+1}_{2n\nu} - \frac{a^{j+1}_{55}(h^{j+1})^3}{12} J^{j+1}_{2n\nu} + \frac{3}{8} a^{j+1}_{55} h^{j+1} \delta_{n\nu}$$

$$- \bar{\alpha}^j_{1n\nu} - \frac{h^j}{2} \left(\frac{n\pi}{\ell}\right)\zeta^j_{1n\nu} - \frac{a^j_{55}(h^j)^3}{12} J^j_{1n\nu} - \frac{3}{8} a^j_{55} h^j \delta_{n\nu} ,$$

$$I_{3nv}^{j} = \bar{\alpha}_{1nv}^{j+1} - \frac{h^{j+1}}{2}\left(\frac{n\pi}{\ell}\right)\zeta_{1nv}^{j+1} - \frac{a_{55}^{j+1}(h^{j+1})^3}{12} J_{1nv}^{j+1} - \frac{1}{8} a_{55}^{j+1} h^{j+1} \delta_{nv} ,$$

$$I_{4nv}^{j} = -\left\{ \bar{\alpha}_{4nv}^{j} + \frac{h^{j}}{2}\left(\frac{n\pi}{\ell}\right)\zeta_{4nv}^{j} + \frac{a_{55}^{j}(h^{j})^3}{12} J_{4nv}^{j} \right\} ,$$

$$I_{5nv}^{j} = \bar{\alpha}_{4nv}^{j+1} - \frac{h^{j+1}}{2}\left(\frac{n\pi}{\ell}\right)\zeta_{4nv}^{j+1} - \frac{a_{55}^{j+1}(h^{j+1})^3}{12} J_{4nv}^{j+1} - \bar{\alpha}_{3nv}^{j}$$

$$- \frac{h^{j}}{2}\left(\frac{n\pi}{\ell}\right)\zeta_{3nv}^{j} - \frac{a_{55}^{j}(h^{j})^3}{12} J_{3nv}^{j} ,$$

$$I_{6nv}^{j} = \bar{\alpha}_{3nv}^{j+1} - \frac{h^{j+1}}{2}\left(\frac{n\pi}{\ell}\right)\zeta_{3nv}^{j+1} - \frac{a_{55}^{j+1}(h^{j+1})^3}{12} J_{3nv}^{j+1} .$$

and where $\delta_{nv}$ is the Kronecker delta.

Just as equation (4.30) holds for any arbitrary value of x, so must equation (4.33). Thus, the second set of continuity conditions were obtained and they are

$$\sum_{v=0}^{\infty} \left\{ I_{1nv}^{j} b_v^{j-1} + I_{2nv}^{j} b_v^{j} + I_{3nv}^{j} b_v^{j+1} + I_{4nv}^{j} d_v^{j-1} + I_{5nv}^{j} d_v^{j} + I_{6nv}^{j} d_v^{j+1} \right\} = 0 \quad (4.35)$$

$$\text{for} \quad j = 1,2,3,\ldots,N-1 ,$$
$$n = 1,2,3,\ldots,\infty .$$

Equations (4.31) and (4.35) may be written in a single matrix equation as follows:

$$\left[\begin{array}{c|c} D_{11} & D_{12} \\ \hline D_{21} & D_{22} \end{array}\right] \left\{\begin{array}{c} \bar{b} \\ \hline \bar{d} \end{array}\right\} = \left\{\begin{array}{c} \bar{0} \\ \hline \bar{0} \end{array}\right\} \quad (4.36)$$

where

$$
[D_{11}] = \begin{bmatrix}
\Gamma_{11} & \Gamma_{12} & & & & & 0 \\
\Gamma_{21} & \Gamma_{22} & \Gamma_{23} & & & & \\
& \Gamma_{32} & \Gamma_{33} & \Gamma_{34} & & & \\
& & \ddots & \ddots & \ddots & & \\
& & \Gamma_{j,j-1} & \Gamma_{j,j} & \Gamma_{j,j+1} & & \\
& 0 & & \ddots & \ddots & \ddots & \\
& & & & & & \Gamma_{N-2,N-1} \\
& & & & & \Gamma_{N-1,N-2} & \Gamma_{N-1,N-1}
\end{bmatrix}; \tag{4.37}
$$

$$
[D_{12}] = \begin{bmatrix}
\overline{F}_{11} & \overline{F}_{12} & & & & & 0 \\
\overline{F}_{21} & \overline{F}_{22} & \overline{F}_{23} & & & & \\
& \overline{F}_{32} & \overline{F}_{33} & \overline{F}_{34} & & & \\
& & \ddots & \ddots & \ddots & & \\
& & \overline{F}_{j,j-1} & \overline{F}_{j,j} & \overline{F}_{j,j+1} & & \\
& 0 & & \ddots & \ddots & \ddots & \\
& & & & & & \overline{F}_{N-2,N-1} \\
& & & & & \overline{F}_{N-1,N-2} & \overline{F}_{N-1,N-1}
\end{bmatrix};
$$

$$
[D_{21}] = \begin{bmatrix}
\overline{H}_{11} & \overline{H}_{12} & & & & & 0 \\
\overline{H}_{21} & \overline{H}_{22} & \overline{H}_{23} & & & & \\
& \overline{H}_{32} & \overline{H}_{33} & \overline{H}_{34} & & & \\
& & \ddots & \ddots & \ddots & & \\
& & \overline{H}_{j,j-1} & \overline{H}_{j,j} & \overline{H}_{j,j+1} & & \\
& 0 & & \ddots & \ddots & \ddots & \\
& & & & & & \overline{H}_{N-2,N-1} \\
& & & & & \overline{H}_{N-1,N-2} & \overline{H}_{N-1,N-1}
\end{bmatrix};
$$

$$[D_{22}] = \begin{bmatrix} \bar{\Lambda}_{11} & \bar{\Lambda}_{12} & & & & & 0 \\ \bar{\Lambda}_{21} & \bar{\Lambda}_{22} & \bar{\Lambda}_{23} & & & & \\ & \bar{\Lambda}_{32} & \bar{\Lambda}_{33} & \bar{\Lambda}_{34} & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \bar{\Lambda}_{j,j-1} & \bar{\Lambda}_{j,j} & \bar{\Lambda}_{j,j+1} & \\ & & & & \ddots & \ddots & \bar{\Lambda}_{N-2,N-1} \\ & 0 & & & & \bar{\Lambda}_{N-1,N-2} & \bar{\Lambda}_{N-1,N-1} \end{bmatrix}$$

$$\{\bar{b}\} = \{\bar{b}^1, \bar{b}^2, \ldots, \bar{b}^j, \ldots, \bar{b}^{N-1}\}^T, \ \{\bar{d}\} = \{\bar{d}^1, \bar{d}^2, \ldots, \bar{d}^j, \ldots, \bar{d}^{N-1}\}^T$$

in which each element of the matrices shown in equation (4.37) is again a matrix. They are represented in the following general form:

$$[\Gamma_{j,j-1}] = \begin{vmatrix} I^j_{111} & I^j_{112} & \cdots & I^j_{11n} & \cdots \\ I^j_{121} & I^j_{122} & \cdots & I^j_{12n} & \cdots \\ \vdots & \vdots & & \vdots & \\ I^j_{1m1} & I^j_{1m2} & \cdots & I^j_{1mn} & \cdots \\ \vdots & \vdots & & \vdots & \end{vmatrix} ; \ [\bar{F}_{j,j-1}] = \begin{vmatrix} I^j_{410} & I^j_{411} & \cdots & I^j_{41n} & \cdots \\ I^j_{420} & I^j_{421} & \cdots & I^j_{42n} & \cdots \\ \vdots & \vdots & & \vdots & \\ I^j_{4m0} & I^j_{4m1} & \cdots & I^j_{4mn} & \cdots \\ \vdots & \vdots & & \vdots & \end{vmatrix} ;$$

(4.38)

$$[\Gamma_{j,j}] = \begin{vmatrix} I^j_{211} & I^j_{212} & \cdots & I^j_{21n} & \cdots \\ I^j_{221} & I^j_{222} & \cdots & I^j_{22n} & \cdots \\ \vdots & \vdots & & \vdots & \\ I^j_{2m1} & I^j_{2m2} & \cdots & I^j_{2mn} & \cdots \\ \vdots & \vdots & & \vdots & \end{vmatrix} ; \quad [\overline{F}_{j,j}] = \begin{vmatrix} I^j_{510} & I^j_{511} & \cdots & I^j_{51n} & \cdots \\ I^j_{520} & I^j_{521} & \cdots & I^j_{52n} & \cdots \\ \vdots & \vdots & & \vdots & \\ I^j_{5m0} & I^j_{5m1} & \cdots & I^j_{5mn} & \cdots \\ \vdots & \vdots & & \vdots & \end{vmatrix} ;$$

$$[\Gamma_{j,j+1}] = \begin{vmatrix} I^j_{311} & I^j_{312} & \cdots & I^j_{31n} & \cdots \\ I^j_{321} & I^j_{322} & \cdots & I^j_{32n} & \cdots \\ \vdots & \vdots & & \vdots & \\ I^j_{3m1} & I^j_{3m2} & \cdots & I^j_{3mn} & \cdots \\ \vdots & \vdots & & \vdots & \end{vmatrix} ; \quad [\overline{F}_{j,j+1}] = \begin{vmatrix} I^j_{610} & I^j_{611} & \cdots & I^j_{61n} & \cdots \\ I^j_{620} & I^j_{621} & \cdots & I^j_{62n} & \cdots \\ \vdots & \vdots & & \vdots & \\ I^j_{6m0} & I^j_{6m1} & \cdots & I^j_{6mn} & \cdots \\ \vdots & \vdots & & \vdots & \end{vmatrix} ;$$

$$[\overline{H}_{j,j+1}] = \begin{vmatrix} L^j_{101} & L^j_{102} & \cdots & L^j_{10n} & \cdots \\ L^j_{111} & L^j_{112} & \cdots & L^j_{11n} & \cdots \\ \vdots & \vdots & & \vdots & \\ L^j_{1m1} & L^j_{1m2} & \cdots & L^j_{1mn} & \cdots \\ \vdots & \vdots & & \vdots & \end{vmatrix} ; \quad [\overline{\Lambda}_{j,j+1}] = \begin{vmatrix} L^j_{400} & L^j_{401} & \cdots & L^j_{40n} & \cdots \\ L^j_{410} & L^j_{411} & \cdots & L^j_{41n} & \cdots \\ \vdots & \vdots & & \vdots & \\ L^j_{4m0} & L^j_{4m1} & \cdots & L^j_{4mn} & \cdots \\ \vdots & \vdots & & \vdots & \end{vmatrix}$$

$$[\bar{H}_{j,j}] = \begin{vmatrix} L^j_{201} & L^j_{202} & \cdots & L^j_{20n} & \cdots \\ L^j_{211} & L^j_{212} & \cdots & L^j_{21n} & \cdots \\ L^j_{221} & L^j_{222} & \cdots & L^j_{22n} & \cdots \\ \vdots & \vdots & & \vdots & \\ L^j_{2m1} & L^j_{2m2} & \cdots & L^j_{2mn} & \cdots \\ \vdots & \vdots & & \vdots & \end{vmatrix} ; \quad [\bar{\Lambda}_{j,j}] = \begin{vmatrix} L^j_{500} & L^j_{501} & \cdots & L^j_{50n} & \cdots \\ L^j_{510} & L^j_{511} & \cdots & L^j_{51n} & \cdots \\ L^j_{520} & L^j_{521} & \cdots & L^j_{52n} & \cdots \\ \vdots & \vdots & & \vdots & \\ L^j_{5m0} & L^j_{5m1} & \cdots & L^j_{5mn} & \cdots \\ \vdots & \vdots & & \vdots & \end{vmatrix} ;$$

$$[\bar{H}_{j,j+1}] = \begin{vmatrix} L^j_{301} & L^j_{302} & \cdots & L^j_{30n} & \cdots \\ L^j_{311} & L^j_{312} & \cdots & L^j_{31n} & \cdots \\ L^j_{321} & L^j_{322} & \cdots & L^j_{32n} & \cdots \\ \vdots & \vdots & & \vdots & \\ L^j_{3m1} & L^j_{3m2} & \cdots & L^j_{3mn} & \cdots \\ \vdots & \vdots & & \vdots & \end{vmatrix} ; \quad [\bar{\Lambda}_{j,j+1}] = \begin{vmatrix} L^j_{600} & L^j_{601} & \cdots & L^j_{60n} & \cdots \\ L^j_{610} & L^j_{611} & \cdots & L^j_{61n} & \cdots \\ L^j_{620} & L^j_{621} & \cdots & L^j_{62n} & \cdots \\ \vdots & \vdots & & \vdots & \\ L^j_{6m0} & L^j_{6m1} & \cdots & L^j_{6mn} & \cdots \\ \vdots & \vdots & & \vdots & \end{vmatrix} ;$$

$$\{\bar{b}^j\} = \begin{Bmatrix} b^j_1 \\ b^j_2 \\ \vdots \\ b^j_m \\ \vdots \end{Bmatrix} ; \quad \{\bar{d}^j\} = \begin{Bmatrix} d^j_0 \\ d^j_1 \\ d^j_2 \\ \vdots \\ d^j_m \\ \vdots \end{Bmatrix} ,$$

for $j = 1,2,3,\ldots,N-1$.

The superscripts $j$ stand for the $j$th contact surface.

For nontrivial solutions of $b_n^j$ and $d_n^j$, $j = 1,2,\ldots,N-1$, and $N = 0,1,2,\ldots,\infty$, the determinant of the coefficient matrix of equation (4.36) must vanish. This yields the frequency equation

$$\left|\begin{array}{c|c} D_{11} & D_{12} \\ \hline D_{21} & D_{22} \end{array}\right| = 0 \; . \tag{4.39}$$

In this chapter, the rotationally symmetric vibrations of an N-layered orthotropic cylindrical shell have been studied, and the frequency equation associated with this shell system has been obtained. Instead of considering the whole laminated shell as one equivalent layer which was used in the past by many authors as discussed in the introduction, each layer was treated as a separate uniform shell of orthotropic material. Different layers may have different thicknesses and/or different elastic properties. By using generalized Fourier series to express the unknown stresses between every two adjacent layers, and requiring the satisfaction of both the continuity conditions and boundary conditions of each individual layer, the free vibration problem of an N-layered orthotropic cylindrical shell has been solved exactly.

This analysis does not consider the rotary inertia of shell layer element about its circumferential centroid, since for thin layers, the corresponding mass moment of inertia is negligible. If the total thickness of multiple layers becomes large, it is conceivable that there could be a contribution from this term. However, Greenspon's work [37] for an isotropic material suggests that even in this case this is probably a higher order effect.

# CHAPTER V

## ROTATIONALLY SYMMETRICAL VIBRATIONS OF CYLINDRICAL SHELLS WITH FIXED EDGES

To illustrate certain aspects of the problem and to better clarify some minor details, the analysis of a cylindrical shell fixed at both edges together with a particular form of solution of the characteristic equation (3.25) will be presented in detail. Numerical results for one-layered and two-layered orthotropic cylindrical shells for various shell dimensions are obtained. The materials considered in these numerical examples are barite and topaz.

### Illustrative Example

The complementary solutions for the particular shell geometry chosen in the example correspond to the case where one has the following inequalities:*

$$q_1^3 + q_2^2 > 0 \ ,$$

and

$$q_3 + q_4 - s_2 < 0 \ .$$

Any other case can be solved without difficulty.** The complementary

---

*This case occurs when the numerical value of $\bar{\lambda}$ is less than 1.68675 for barite and 2.05608 for topaz, in which the cylinder has dimensions $h = 0.06$ in., $R = 6$ in. and $\ell = 2.4$ in.

**The two other most frequent cases for these two kinds of material are also given in Appendix D.

solution is therefore given by

$$W_H^j = K_1^j \cos \beta_o^j x + e^{\alpha_1^j x}(K_2^j \cos \beta_1^j x + K_3^j \sin \beta_1^j x) + K_4^j \sin\beta_o^j x \quad (5.1)$$

$$+ e^{-\alpha_1^j x} (K_5^j \cos \beta_1^j x + K_6^j \sin \beta_1^j x)$$

where $\alpha_1^j$, $\beta_1^j$ and $\beta_o^j$ are given in equation (B.7) for a jth-layered shell.

From equations (4.11), (4.12) and (4.13), the general solutions for $W^j$, $U^j$ and $\Phi^j$ are

$$W^j = K_1^j \cos \beta_o^j x + e^{\alpha_1^j x}(K_2^j \cos \beta_1^j x + K_3^j \sin \beta_1^j x) + K_4^j \sin\beta_o^j x \quad (5.2)$$

$$+ e^{-\alpha_1^j x} (K_5^j \cos \beta_1^j x + K_6^j \sin \beta_1^j x)$$

$$+ \sum_{n=0}^{\infty} T_n^j(A_{1n}^j b_n^j + A_{2n}^j b_n^{j-1} + A_{3n}^j d_n^j + A_{4n}^j d_n^{j-1})\cos \frac{n\pi x}{\ell},$$

$$U^j = \eta_1^{*j}(K_1^j \sin \beta_o^j x - K_4^j \cos \beta_o^j x) + e^{\alpha_1^j x}[K_2^j(\eta_2^j \cos \beta_1^j x \quad (5.3)$$

$$- \eta_3^j \sin \beta_1^j x) + K_3^j(\eta_3^j \cos \beta_1^j x + \eta_2^j \sin \beta_1^j x)]$$

$$+ e^{-\alpha_1^j x}[K_5^j(-\eta_2^j \cos \beta_1^j x - \eta_3^j \sin \beta_1^j x) + K_6^j(\eta_3^j \cos\beta_1^j x$$

$$- \eta_2^j \sin \beta_1^j x)] + \sum_{n=1}^{\infty} (\bar{a}_{1n}^j b_n^j + \bar{a}_{2n}^j b_n^{j-1} + \bar{a}_{3n}^j d_n^j + \bar{a}_{4n}^j d_n^{j-1})\sin \frac{n\pi x}{\ell}$$

and

$$\Phi^j = \eta_4^{*j}(K_1^j \sin\beta_o^j x - K_4^j \cos\beta_o^j x) + e^{\alpha_1^j x}[K_2^j(\eta_5^j \cos\beta_1^j x \tag{5.4}$$

$$- \eta_6^j \sin\beta_1^j x) + K_3^j(\eta_6^j \cos\beta_1^j x + \eta_5^j \sin\beta_1^j x)] + e^{-\alpha_1^j x}[K_5^j(-\eta_5^j \cdot$$

$$\cos\beta_1^j x - \eta_6^j \sin\beta_1^j x) + K_6^j(\eta_6^j \cos\beta_1^j x - \eta_5^j \sin\beta_1^j x)]$$

$$+ \sum_{n=1}^{\infty}(\bar{g}_{1n}^j b_n^j + \bar{g}_{2n}^j b_n^{j-1} + \bar{g}_{3n}^j d_n^j + \bar{g}_{4n}^j d_n^{j-1}) \sin\frac{n\pi x}{\ell}$$

where $\eta_1^{*j}$, $\eta_4^{*j}$, $\eta_2^j$, $\eta_3^j$, $\eta_5^j$ and $\eta_6^j$ are given in equation (B.12).

The matrice $[\bar{\xi}^j]$ and $\{\bar{p}^j\}$ in equation (4.17a) are

$$[\bar{\xi}]^j = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & \alpha_1 & \beta_1 & \beta_o & -\alpha_1 & \beta_1 \\ 0 & \eta_2 & \eta_3 & -\eta_1^* & -\eta_2 & \eta_3 \\ \cos\beta_o\ell & e^{\alpha_1\ell}\cdot\cos\beta_1\ell & e^{\alpha_1\ell}\cdot\sin\beta_1\ell & \sin\beta_o\ell & e^{-\alpha_1\ell}\cdot\cos\beta_1\ell & e^{-\alpha_1\ell}\cdot\sin\beta_1\ell \\ -\beta_o\sin\beta_o\ell & \eta_{16} & \eta_{17} & \beta_o\cos\beta_o\ell & \eta_{16}^* & \eta_{17}^* \\ \eta_1^*\sin\beta_o\ell & \eta_{18} & \eta_{19} & -\eta_1^*\cos\beta_o\ell & \eta_{18}^* & \eta_{19}^* \end{bmatrix}^{j\,-1} \tag{5.5a}$$

and

$$\{\bar{p}^j\} = \begin{Bmatrix} -\sum_{n=0}^{\infty} \bar{K}_n^j \\ 0 \\ 0 \\ -\sum_{n=0}^{\infty} (-1)^n \bar{K}_n^j \\ 0 \\ 0 \end{Bmatrix} \tag{5.5b}$$

where $\overline{K}_n^j = T_n^j(A_{1n}^j b_n^j + A_{2n}^j b_n^{j-1} + A_{3n}^j d_n^j + A_{4n}^j d_n^{j-1})$ , and where the superscript j of each matrix stands for jth layered shell.

The functions $\sin \beta_o^j x$, $\cos \beta_o^j x$, $e^{\alpha_1^j x} \cos \beta_1^j x$, $e^{\alpha_1^j x} \sin \beta_1^j x$, $e^{-\alpha_1^j x} \cos \beta_1^j x$ and $e^{-\alpha_1^j x} \sin \beta_1^j x$, expressed in terms of their Fourier cosine series expansions over the interval $(0, \ell)$, are

$$\cos \beta_o^j x = \sum_{n=0}^{\infty} t_{1n}^j \cos \frac{n\pi x}{\ell} , \tag{5.7}$$

$$e^{\alpha_1^j x} \cos \beta_1^j x = \sum_{n=0}^{\infty} t_{2n}^j \cos \frac{n\pi x}{\ell} ,$$

$$e^{\alpha_1^j x} \sin \beta_1^j x = \sum_{n=0}^{\infty} t_{3n}^j \cos \frac{n\pi x}{\ell} ,$$

$$\sin \beta_o^j x = \sum_{n=0}^{\infty} t_{4n}^j \cos \frac{n\pi x}{\ell} ,$$

$$e^{-\alpha_1^j x} \cos \beta_1^j x = \sum_{n=0}^{\infty} t_{5n}^j \cos \frac{n\pi x}{\ell} ,$$

$$e^{-\alpha_1^j x} \sin \beta_1^j x = \sum_{n=0}^{\infty} t_{6n}^j \cos \frac{n\pi x}{\ell} ,$$

where

$$t_{10}^j = \frac{\sin \beta_o \ell}{\beta_o \ell} , \tag{5.8}$$

$$t_{20}^{j} = \frac{1}{\ell}\left\{\frac{e^{\alpha_1^j \ell}[\alpha_1^j \cos\beta_1^j\ell + \beta_1^j \sin\beta_1^j\ell] - \alpha_1^j}{\alpha_1^{j^2} + \beta_1^{j^2}}\right\},$$

$$t_{30}^{j} = \frac{1}{\ell}\left\{\frac{e^{\alpha_1^j \ell}[\alpha_1^j \sin\beta_1^j\ell - \beta_1^j \cos\beta_1^j\ell] + \beta_1^j}{\alpha_1^{j^2} + \beta_1^{j^2}}\right\},$$

$$t_{40}^{j} = \frac{1 - \cos\beta_o\ell}{\ell\beta_o},$$

$$t_{50}^{j} = \frac{1}{\ell}\left\{\frac{e^{-\alpha_1^j \ell}[-\alpha_1^j \cos\beta_1^j\ell + \beta_1^j \sin\beta_1^j\ell] + \alpha_1^j}{\alpha_1^{j^2} + \beta_1^{j^2}}\right\},$$

$$t_{60}^{j} = \frac{1}{\ell}\left\{\frac{e^{-\alpha_1^j \ell}[-\alpha_1^j \sin\beta_1^j\ell - \beta_1^j \cos\beta_1^j\ell] + \beta_1^j}{\alpha_1^{j^2} + \beta_1^{j^2}}\right\},$$

$$t_{1n}^{j} = \frac{(-1)^n\, 2\beta_o^j \sin\beta_o^j\ell}{\ell[\beta_o^{j^2} - (\frac{n\pi}{\ell})^2]},$$

$$t_{2n}^{j} = \frac{1}{\ell}\left\{\frac{(-1)^n e^{\alpha_1^j \ell}[\alpha_1^j \cos\beta_1^j\ell + (\beta_1^j + \frac{n\pi}{\ell}) \sin\beta_1^j\ell] - \alpha_1^j}{(\alpha_1^j)^2 + (\beta_1^j + \frac{n\pi}{\ell})^2}\right.$$

$$\left. + \frac{(-1)^n e^{\alpha_1^j \ell}[\alpha_1^j \cos\beta_1^j\ell + (\beta_1^j - \frac{n\pi}{\ell}) \sin\beta_1^j\ell] - \alpha_1^j}{(\alpha_1^j)^2 + (\beta_1^j - \frac{n\pi}{\ell})^2}\right\},$$

$$t_{3n}^{j} = \frac{1}{\ell}\left\{\frac{(-1)^n\, e^{\alpha_1^j \ell}[\alpha_1^j \sin \beta_1^j \ell - (\beta_1^j + \frac{n\pi}{\ell}) \cos \beta_1^j \ell] + \beta_1^j + \frac{n\pi}{\ell}}{(\alpha_1^j)^2 + (\beta_1^j + \frac{n\pi}{\ell})^2}\right.$$

$$\left. + \frac{(-1)^n\, e^{\alpha_1^j \ell}[\alpha_1^j \sin \beta_1^j - (\beta_1^j - \frac{n\pi}{\ell}) \cos \beta_1^j \ell] + \beta_1^j - \frac{n\pi}{\ell}}{(\alpha_1^j)^2 + (\beta_1^j - \frac{n\pi}{\ell})^2}\right\},$$

$$t_{4n}^{j} = \frac{2[\beta_o^j + (-1)^{n+1}\, \beta_o^j \cos \beta_o^j \ell]}{\ell[\beta_o^{j^2} - (\frac{n\pi}{\ell})^2]} \quad ,$$

$$t_{5n}^{j} = \frac{1}{\ell}\left\{\frac{(-1)^n\, e^{-\alpha_1^j \ell}[-\alpha_1^j \cos \beta_1^j \ell + (\beta_1^j + \frac{n\pi}{\ell}) \sin \beta_1^j \ell] + \alpha_1^j}{(\alpha_1^j)^2 + (\beta_1^j + \frac{n\pi}{\ell})^2}\right.$$

$$\left. + \frac{(-1)^n\, e^{-\alpha_1^j \ell}[-\alpha_1^j \cos \beta_1^j \ell + (\beta_1^j - \frac{n\pi}{\ell}) \sin \beta_1^j \ell] + \alpha_1^j}{(\alpha_1^j)^2 + (\beta_1^j - \frac{n\pi}{\ell})^2}\right\},$$

$$t_{6n}^{j} = \frac{1}{\ell}\left\{\frac{(-1)^n\, e^{-\alpha_1^j \ell}[-\alpha_1^j \sin \beta_1^j \ell - (\beta_1^j + \frac{n\pi}{\ell}) \cos \beta_1^j \ell] + \beta_1^j + \frac{n\pi}{\ell}}{(\alpha_1^j)^2 + (\beta_1^j + \frac{n\pi}{\ell})^2}\right.$$

$$\left. + \frac{(-1)^n\, e^{-\alpha_1^j \ell}[-\alpha_1^j \sin \beta_1^j \ell - (\beta_1^j - \frac{n\pi}{\ell}) \cos \beta_1^j \ell] + \beta_1^j - \frac{n\pi}{\ell}}{(\alpha_1^j)^2 + (\beta_1^j - \frac{n\pi}{\ell})^2}\right\}.$$

Similarly, the Fourier sine expansions are

$$\cos \beta_o^j x = \sum_{n=1}^{\infty} r_{1n}^j \sin \frac{n\pi x}{\ell} \qquad (5.9)$$

$$e^{\alpha_1^j x} \cos \beta_1^j x = \sum_{n=1}^{\infty} r_{2n}^j \sin \frac{n\pi x}{\ell} ,$$

$$e^{\alpha_1^j x} \sin \beta_1^j x = \sum_{n=1}^{\infty} r_{3n}^j \sin \frac{n\pi x}{\ell} ,$$

$$\sin \beta_o^j x = \sum_{n=1}^{\infty} r_{4n}^j \sin \frac{n\pi x}{\ell} ,$$

$$e^{-\alpha_1^j x} \cos \beta_1^j x = \sum_{n=1}^{\infty} r_{5n}^j \sin \frac{n\pi x}{\ell} ,$$

$$e^{-\alpha_1^j x} \sin \beta_1^j x = \sum_{n=1}^{\infty} r_{6n}^j \sin \frac{n\pi x}{\ell} ,$$

where

$$r_{1n}^j = -\frac{2(\frac{n\pi}{\ell})[1 - (-1)^n \cos \beta_o^j \ell]}{\ell[{\beta_o^j}^2 - (\frac{n\pi}{\ell})^2]} , \tag{5.10}$$

$$r_{2n}^j = \frac{1}{\ell}\left\{\frac{(-1)^n e^{\alpha_1^j \ell}[\alpha_1^j \sin \beta_1^j \ell - (\beta_1^j + \frac{n\pi}{\ell}) \cos \beta_1^j \ell] + \beta_1^j + \frac{n\pi}{\ell}}{(\alpha_1^j)^2 + (\beta_1^j + \frac{n\pi}{\ell})^2}\right.$$

$$\left. - \frac{(-1)^n e^{\alpha_1^j \ell}[\alpha_1^j \sin \beta_1^j \ell - (\beta_1^j - \frac{n\pi}{\ell}) \cos \beta_1^j \ell] + \beta_1^j - \frac{n\pi}{\ell}}{(\alpha_1^j)^2 + (\beta_1^j - \frac{n\pi}{\ell})^2}\right\} ,$$

$$r_{3n}^j = \frac{1}{\ell}\left\{\frac{-(-1)^n e^{\alpha_1^j \ell}[\alpha_1^j \cos \beta_1^j \ell + (\beta_1^j + \frac{n\pi}{\ell}) \sin \beta_1^j \ell] + \alpha_1^j}{(\alpha_1^j)^2 + (\beta_1^j + \frac{n\pi}{\ell})^2}\right.$$

$$+\frac{(-1)^n e^{\alpha_1^j \ell}[\alpha_1^j \cos \beta_1^j \ell + (\beta_1^j - \frac{n\pi}{\ell}) \sin \beta_1^j \ell] - \alpha_1^j}{(\alpha_1^j)^2 + (\beta_1^j - \frac{n\pi}{\ell})^2}\Bigg\} ,$$

$$r_{4n}^j = \frac{2(-1)^n(\frac{n\pi}{\ell}) \sin \beta_o^j \ell}{\ell[\beta_o^{j\,2} - (\frac{n\pi}{\ell})^2]} ,$$

$$r_{5n}^j = \frac{1}{\ell}\Bigg\{\frac{(-1)^n e^{-\alpha_1^j \ell}[-\alpha_1^j \sin \beta_1^j \ell - (\beta_1^j + \frac{n\pi}{\ell}) \cos \beta_1^j] + \frac{n\pi}{\ell} + \beta_1^j}{(\alpha_1^j)^2 + (\beta_1^j + \frac{n\pi}{\ell})^2}$$

$$-\frac{(-1)^n e^{-\alpha_1^j \ell}[-\alpha_1^j \sin \beta_1^j \ell - (\beta_1^j - \frac{n\pi}{\ell}) \cos \beta_1^j \ell] + \beta_1^j - \frac{n\pi}{\ell}}{(\alpha_1^j)^2 + (\beta_1^j - \frac{n\pi}{\ell})^2}\Bigg\},$$

$$r_{6n}^j = \frac{1}{\ell}\Bigg\{\frac{-(-1)^n e^{-\alpha_1^j \ell}[-\alpha_1^j \cos \beta_1^j \ell + (\beta_1^j + \frac{n\pi}{\ell}) \sin \beta_1^j \ell] - \alpha_1^j}{(\alpha_1^j)^2 + (\beta_1^j + \frac{n\pi}{\ell})^2}$$

$$+\frac{(-1)^n e^{-\alpha_1^j \ell}[-\alpha_1^j \cos \beta_1^j \ell + (\beta_1^j - \frac{n\pi}{\ell}) \sin \beta_1^j \ell] + \alpha_1^j}{(\alpha_1^j)^2 + (\beta_1^j - \frac{n\pi}{\ell})^2}\Bigg\} .$$

The substitution of equations (5.7) and (5.9) into equations (5.2), (5.3) and (5.4) results in the following expressions for the shell displacement coordinate functions and shear stress function in terms of the unknown Fourier loading coefficients:

$$W^j = \sum_{n=0}^{\infty}\Bigg\{ K_1^j\, t_{1n}^j + K_2^j\, t_{2n}^j + K_3^j\, t_{3n}^j + K_4^j\, t_{4n}^j + K_5^j\, t_{5n}^j + K_6^j\, t_{6n}^j \quad (5.11)$$

$$+ T_n^j(A_{1n}^j b_n^j + A_{2n}^j b_n^{j-1} + A_{3n}^j d_n^j + A_{4n}^j d_n^{j-1})\Bigg\} \cos \frac{n\pi x}{\ell} ,$$

$$U^j = \sum_{n=1}^{\infty}\Big\{K_1^j\ \eta_1^{*j} r_{4n}^j + K_2^j(\eta_2^j r_{2n}^j - \eta_3^j r_{3n}^j) + K_3^j(\eta_3^j r_{2n}^j + \eta_2^j r_{3n}^j) \tag{5.12}$$

$$- K_4^j \eta_1^{*j} r_{1n}^j + K_5^j(-\eta_2^j r_{5n}^j - \eta_3^j r_{6n}^j) + K_6^j(\eta_3^j r_{5n}^j - \eta_2^j r_{6n}^j)$$

$$+ \bar{a}_{1n}^j b_n^j + \bar{a}_{2n}^j b_n^{j-1} + \bar{a}_{3n}^j d_n^j + \bar{a}_{4n}^j d_n^{j-1}\Big\} \sin\frac{n\pi x}{\ell}$$

and

$$\Phi^j = \sum_{n=1}^{\infty}\Big\{K_1^j \eta_4^{*j} r_{4n}^j + K_2^j(\eta_5^j r_{2n}^j - \eta_6^j r_{3n}^j) + K_3^j(\eta_6^j r_{2n}^j + \eta_5^j r_{3n}^j) \tag{5.13}$$

$$- K_4^j \eta_4^{*j} r_{1n}^j - K_5^j(\eta_5^j r_{5n}^j + \eta_6^j r_{6n}^j) + K_6^j(\eta_6^j r_{5n}^j - \eta_5^j r_{6n}^j)$$

$$+ \bar{g}_{1n}^j b_n^j + \bar{g}_{2n}^j b_n^{j-1} + \bar{g}_{3n}^j d_n^j + \bar{g}_{4n}^j d_n^{j-1}\Big\} \sin\frac{n\pi x}{\ell} .$$

From equation (4.17a), in conjunction with equation (5.5), the constants of integration $K_\mu^j$, $\mu = 1,2,3,\ldots,6$, when expressed in terms of $b_n^j$, $b_n^{j-1}$, $d_n^j$ and $d_n^{j-1}$, have the following forms:

$$K_\mu^j = -\sum_{\nu=0}^{\infty} T_\nu^j\,[\xi_{\mu 1}^j + (-1)^\nu\ \xi_{\mu 4}^j](A_{1\nu}^j b_\nu^j + A_{2\nu}^j b_\nu^{j-1} + A_{3\nu}^j d_\nu^j + A_{4\nu}^j d_\nu^{j-1}) \tag{5.15}$$

$$\text{for } \mu = 1,2,3,\ldots,6 .$$

The coefficients $\zeta_{\mu n\nu}^j$, $\bar{\alpha}_{\mu n\nu}^j$ and $J_{\mu n\nu}^j$ of equations (4.23), (4.24) and (4.25) are given by

$$\zeta_{\mu n\nu}^j = A_{\mu\nu}^j \zeta_{n\nu}^{*j} , \tag{5.16}$$

$$\bar{\alpha}_{\mu n\nu}^j = A_{\mu\nu}^j \alpha_{n\nu}^{*j} + \bar{a}_{\mu n}^j \delta_{n\nu} ,$$

$$J^j_{\mu n\nu} = A^j_{\mu\nu} J^{*j}_{n\nu} + \bar{g}^j_{\mu n}\,\delta_{n\nu} \;,$$

$$\text{for } \mu = 1,2,3,4,$$
$$n = 0,1,2,\ldots,\infty\;,$$
$$\nu = 0,1,2,\ldots,\infty\;,$$

where $\bar{g}^j_{\mu n}$ and $\bar{a}^j_{\mu n}$, $\mu = 1,2,3,4$, are defined in equation (4.14), and $A^j_{\mu n}$ are defined in equation (4.7), and where

$$\zeta^{*j}_{n\nu} = T^j_\nu \left\{ \delta_{n\nu} - \sum_{\mu=1}^{6} [\xi^j_{\mu 1} + (-1)^\nu\, \xi^j_{\mu 4}] t^j_{n\mu} \right\} \;, \tag{5.17}$$

$$\begin{aligned} \alpha^{*j}_{n\nu} = - T^j_\nu \Big\{ & \eta^{*j}_1 r^j_{4n}[\xi^j_{11} + (-1)^\nu \xi^j_{14}] + (\eta^j_2 r^j_{2n} - \eta^j_3 r^j_{3n})[\xi^j_{21} + (-1)^\nu \xi^j_{24}] \\ & + (\eta^j_3 r^j_{2n} + \eta^j_2 r^j_{3n})[\xi^j_{31} + (-1)^\nu \xi^j_{34}] - \eta^{*j}_1 r^j_{1n}[\xi^j_{41} + (-1)^\nu \xi^j_{44}] \\ & - (\eta^j_2 r^j_{5n} + \eta^j_3 r^j_{6n})[\xi^j_{51} + (-1)^\nu \xi^j_{54}] + (\eta^j_3 r^j_{5n} - \eta^j_2 r^j_{6n})[\xi^j_{61} + (-1)^\nu \xi^j_{64}] \Big\}, \end{aligned}$$

$$\begin{aligned} J^{*j}_{n\nu} = -T^j_\nu \Big\{ & \eta^{*j}_4 r^j_{4n}[\xi^j_{11} + (-1)^\nu \xi^j_{14}] + (\eta^j_5 r^j_{2n} - \eta^j_6 r^j_{3n})[\xi^j_{21} + (-1)^\nu \xi^j_{24}] \\ & + (\eta^j_6 r^j_{2n} + \eta^j_5 r^j_{3n})[\xi^j_{31} + (-1)^\nu \xi^j_{34}] - \eta^{*j}_4 r^j_{1n}[\xi^j_{41} + (-1)^\nu \xi^j_{44}] \\ & - (\eta^j_5 r^j_{5n} + \eta^j_6 r^j_{6n})[\xi^j_{51} + (-1)^\nu \xi^j_{54}] + (\eta^j_6 r^j_{5n} - \eta^j_5 r^j_{6n})[\xi^j_{61} + (-1)^\nu \xi^j_{64}] \Big\}. \end{aligned}$$

The frequency equation of the layered shell is

$$\left| \begin{array}{c|c} D_{11} & D_{12} \\ \hline D_{21} & D_{22} \end{array} \right| = 0 \tag{5.18}$$

where the submatrices $[D_{ik}]$ are defined in equations (4.37) and (4.38).

## Frequency Equation for a Cylindrical Shell Having One Orthotropic Layer

For the case of free vibration of a shell having one orthotropic layer, there are no contact surfaces or external loads. The frequency equation is given by

$$|\overline{D}| = 0 , \tag{3.34}$$

where $|\overline{D}|$ has different forms depending on the forms of the roots of the characteristic equation (3.25) (which are given in Appendix C and have been discussed in detail in the last section of Chapter III).

## Frequency Equation for a Cylindrical Shell Having Two Orthotropic Layers

For the case of a two-layered orthotropic cylindrical shell, there is only one contact surface; equations (4.31) and (4.35) thus reduce to

$$\sum_{\nu=0}^{\infty} \left\{ L^1_{2n\nu} b^1_\nu + L^1_{5n\nu} d^1_\nu \right\} = 0 , \quad \text{for } n = 0,1,2,\ldots,\infty , \tag{5.19}$$

and

$$\sum_{\nu=0}^{\infty} \left\{ I^1_{2n\nu} b^1_\nu + I^1_{5n\nu} d^1_\nu \right\} = 0 , \quad \text{for } n = 1,2,3,\ldots,\infty , \tag{5.20}$$

where $b^1_0 = 0$. The frequency equation becomes

$$\left|\begin{array}{c|c} D_{11} & D_{12} \\ \hline D_{21} & D_{22} \end{array}\right| = 0 \ , \tag{5.21}$$

which has to be solved simultaneously with

$$\omega^2 = \frac{B_{11}^{(1)}}{\rho(1)} \ \bar{\lambda}^{(1)^2} = \frac{B_{11}^{(2)}}{\rho(2)} \bar{\lambda}^{(2)^2} \tag{5.22}$$

for the eigenvalues $\bar{\lambda}^{(1)}$ and $\bar{\lambda}^{(2)}$ which are related to the coupled frequency parameter of the first and the second layered shell by equation (5.22).

The sub-matrices in the frequency equation have the following forms:

$$[D_{11}] = [\Gamma_{11}] \ ,$$

$$[D_{22}] = [\bar{F}_{11}] \ ,$$

$$[D_{21}] = [\bar{H}_{11}] \ ,$$

$$[D_{22}] = [\bar{\Lambda}_{11}]$$

where $[\Gamma_{11}]$, $[\bar{F}_{11}]$, $[\bar{H}_{11}]$ and $[\bar{\Lambda}_{11}]$ are given in equation (4.38) for $j = 1$.

## Numerical Examples

For quantitative illustrative purposes, a fixed-ended cylindrical shell having one and two orthotropic layers with various geometrical dimensions are investigated. The materials considered in the numerical examples are barite and topaz. Their material constants shown in Table 1 are obtained from Trent and Stone [16], and Clark [17].

Table 1. Some Constants of Barite and Topaz

(a) Elastic Constants
($10^{-8}$ $in^2/lb$)

| | $a_{11}$ | $a_{12}$ | $a_{13}$ | $a_{22}$ | $a_{23}$ | $a_{33}$ | $a_{44}$ | $a_{55}$ | $a_{66}$ |
|---|---|---|---|---|---|---|---|---|---|
| Barite | 12.69 | -6.52 | -1.85 | 11.97 | -1.88 | 7.56 | 57.45 | 24.02 | 25.17 |
| Topaz | 3.05 | -0.95 | -0.59 | 2.43 | -0.04 | 2.65 | 6.35 | 5.19 | 5.26 |

(b) Density
($lb\text{-}sec^2/in^4$)

| | |
|---|---|
| Barite | $0.417 \times 10^{-3}$ |
| Topaz | $0.331 \times 10^{-3}$ |

(c) Value of $B_{11}$ Used in Equation (3.10a)
($lb/in^2$)

| | |
|---|---|
| Barite | $12.4975 \times 10^6$ |
| Topaz | $39.3677 \times 10^6$ |

Since both longitudinal and transverse inertia are included in the analysis, a frequency pair results for each mode. This is similar in character to what is found from the Timoshenko beam equations. It should be noted here that for a "w" mode, say, the corresponding "u" contribution for a given frequency pair will be different. Similar to the Timoshenko beam, the difference between the frequencies is generally quite large. This is true for a short shell. Here, the higher branch frequency is sufficiently removed as not to be of practical interest. But for long shells the two frequency values become closer. In this case it may be important in finding a forced response by these modes to account for both branches. Numerical results presented in this thesis represent in all cases the low frequency branch.

In view of considerable difficulties connected with the determination of the exact roots of a transcendental frequency equation, a numerical procedure is devised which can accurately determine the consecutive frequency parameters $\bar{\lambda}$. The frequency parameter was found by starting from some initially estimated frequency parameter $\bar{\lambda}$, and iterating to find the value of $\bar{\lambda}$ for which the determinant of the frequency equation goes to zero.

<u>One Layered Shell</u>. Since the shell geometry and the edge supporting conditions are symmetric about the middle section of the shell as indicated in Chapter III, it is convenient for numerical computations to place the origin at the mid-point of the cylinder. By doing so, the natural frequencies of the cylindrical shells can always be divided into two groups. One of these corresponds to symmetric modal shapes while the other corresponds to the antisymmetric modes of vibration.

All pertinent parameters and forms of general solutions related to the frequency equation for a fixed-ended shell having one orthotropic layer are presented in Appendix C in detail. Different forms of the roots of equation (3.25) give different expressions for the displacement coordinate functions and their corresponding frequency equations. For example, if a shell has dimensions $R = 6$ inches, $R/h = 100$, and $\ell/R = 0.4$, the frequency equation changes from equation (C.7) to (C.18) for symmetric vibration and from equation (C.22) to (C.33) for antisymmetric vibration when the value of frequency parameter $\bar{\lambda}$ is greater than 1.68675 for barite and 2.05608 for topaz, respectively; and from equation (C.18) to (C.16) for symmetric vibration and from equation (C.33) to (C.31) for antisymmetric vibration when $\bar{\lambda}$ is greater than 2.52130 for barite and 2.34396 for topaz, respectively.

The lowest seven frequencies of both barite and topaz are obtained for $R/h = 25$ and 100, and $R = 6$ inches. They are given in Tables 2 to 5 and plotted in Figures 2 to 5, respectively. In order to determine the effect of shear deformation, the same problem is considered in Appendix E excluding the shear deformation. Two cases are presented there, the first case includes longitudinal inertia and the second case excludes longitudinal inertia. Four frequencies corresponding to $m = 2,3,4$ and 5 obtained according to the present analysis are compared with the results obtained by employing the Kirchhoff assumptions and neglecting the longitudinal inertia in Tables 6 to 11 for various shell dimensions.

For a fixed value of $R$ and $\ell/R$, Figures 2 to 5 indicate that frequency increases monotonically as the number of axial waves increases.

For a fixed value of R and h/R, the frequencies for $m = 1,2,3,\ldots,$ all decrease very rapidly for small values of $\ell/R$, and then gradually approach the membrane frequency (frequency of membrane cylinder with same radius) with increasing $\ell/R$. When the ratio $\ell/R$ increases further, these frequencies decrease rapidly and then approach one another again. The longer the shell, the closer the frequencies are to those of the adjacent axial modes.

From Figures 6 to 9, it is clear that for R = 6 inches, and $\ell/R = 0.5$ and 1.0, all the frequencies vary with the ratio R/h. When R/h increases, all the frequencies, regardless of the axial half-wave number, gradually approach the same value. This indicates that when R/h becomes large, the problem reduces to the problem of the rotationally symmetric vibration of the membrane cylinder, i.e. becomes independent of thickness and the number of axial waves. There is only one natural frequency (membrane frequency) for this limiting case, namely

$$\bar{\lambda}^2 = E^* \left(\frac{1}{R}\right)^2 \qquad (5.28)$$

where

$$\bar{\lambda}^2 = \frac{\rho\omega^2}{B_{11}} \; ; \; E^* = \frac{E_2}{B_{11}} \; . \qquad (5.29)$$

If the Kirchhoff assumptions are used and longitudinal inertia is neglected, then from Figures 2 to 5, for certain fixed shell dimensions, the lowest frequency always occurs for a mode having one axial half-wave. All the frequencies, regardless of the number of axial half-waves, are close together and gradually approach the membrane frequency for the same cylindrical shell. However, when longitudinal inertia and transverse shear effects are included, and the axial constraint conditions ($u = 0$ at edges) are imposed, the lowest frequency does not always correspond to a mode having one axial half-wave. For very short shells, the minimum frequency of rotationally symmetric vibration will occur for $m = 1$. In all the cases calculated here, the lowest frequencies correspond to $m = 2$ for large $\ell/R$. For fixed R and R/h, the value of $\ell/R$ for which this change takes place depends upon the material properties and the value of h/R. For R = 6 inches and R/h = 100, this change occurs when $\ell/R > 0.34$ for barite and $\ell/R > 0.430$ for topaz; similarly for R = 6 inches and R/h = 25, this change occurs when $\ell/R > 0.68$ for barite and $\ell/R > 0.85$ for topaz. It is clear that the value of $\ell/R$ for which this change takes place increases when the thickness of the shell increases. A shell, which is constructed of isotropic material, has been used to check whether this change occurred only for orthotropic cylindrical shells. It was found that this change occurred also for a shell constructed of isotropic material.

For small values of $\ell/R$, the difference between the frequencies obtained by the present analysis and the one based on the Kirchhoff hypothesis (also neglecting the longitudinal inertia), is small. The results obtained in the present analysis are slightly lower than those obtained by using the hypothesis of nondeformable normals. The difference becomes significant when $\ell/R$ becomes large. For instance, when R = 6 inches, $\ell/R = 10.0$, and $m = 2$, the frequencies obtained by using the present analysis are about one quarter of the one obtained by using the Kirchhoff assumptions and neglecting the longitudinal inertia as shown in Figures 11 and 12. This large change in the frequency is due to the effects of the longitudinal inertia, transverse shear deformation and axial constraints. However, when longitudinal inertia of the shell is included in the classical analysis, from Figure 4, it is clear that the effect of the shear deformation is almost negligible when the length of the shell becomes large. But for the short shell, the effect of the shear deformation becomes significant even the thickness of the orthotropic shell is relatively thin ($R/h = 100$). For instance, for $m = 2$ and $\ell/R = 0.5$ (see Figure 4), the difference in the frequencies obtained with and without the shear deformation is about 40 percent.

In order to obtain the mode shapes corresponding to each natural frequency, the frequency parameter obtained is then substituted into equation (3.33). The equations may then be normalized with respect to $K_1$, with the result

$$\begin{bmatrix} D_{12} & D_{13} & \cdots & D_{16} \\ D_{22} & D_{23} & \cdots & D_{26} \\ \vdots & \vdots & & \vdots \\ D_{52} & D_{53} & \cdots & D_{56} \end{bmatrix} \begin{Bmatrix} K_2/K_1 \\ K_3/K_1 \\ \vdots \\ K_6/K_1 \end{Bmatrix} = - \begin{Bmatrix} D_{11} \\ D_{21} \\ \vdots \\ D_{51} \end{Bmatrix} \tag{5.30}$$

or

$$[\bar{D}^*] \begin{Bmatrix} K_2^* \\ K_3^* \\ \vdots \\ K_6^* \end{Bmatrix} = - \left\{ \bar{P}_1^* \right\} . \tag{5.31}$$

The eigenvector was obtained by the multiplication of the inverse of matrix $[\bar{D}^*]$ with $-\{\bar{p}_1^*\}$, or

$$\{\bar{K}^*\} = -[\bar{D}^*]^{-1}\{\bar{p}_1^*\} . \qquad (5.32)$$

The result of the substitution of the eigenvector

$$\{K_1, K_2, K_3, K_4, K_5, K_6\} = \{1, K_2^*, K_3^*, K_4^*, K_5^*, K_6^*\} \qquad (5.33)$$

into equation (3.29) yields the equation from which the mode shape corresponding to each particular natural frequency is computed.

The half range of the first three normalized mode shapes of the shell which was made of topaz and had the dimensions R = 6 inches, $R/\ell$ = 2.0 and R/h = 25 are given in Figure 12. It is seen that the first and the third modes for transverse displacement (w) are symmetric about the mid-section and have one half-wave and three half-waves, respectively. The second mode for transverse displacement is anti-symmetric and has two half-waves. It is also clear from Figure 12 and similar figures for other geometries that the first and the third modes for axial displacement (u) are antisymmetric about the mid-section, while the second mode for axial displacement is symmetric. The longitudinal displacement u is much smaller than the normal displacement w (about the order of one-tenth), and this was also observed for various geometries.

Two Layered Shell. The general expression involved in the frequency equation and the numerical computations for the investigation of a multi-layered shell are substantially more complicated than that of a

Table 2. Frequencies of Single-layered Shell Made of Barite with R = 6 inches and R/h = 100

| $\ell/R$ | $\bar{\lambda}_1$ | $\bar{\lambda}_2$ | $\bar{\lambda}_3$ | $\bar{\lambda}_4$ | $\bar{\lambda}_5$ | $\bar{\lambda}_6$ | $\bar{\lambda}_7$ |
|---|---|---|---|---|---|---|---|
| 0.2 | 3.53510 | 9.04963 | 17.43952 | 28.86867 | 43.03283 | 60.20635 | 80.16836 |
| 0.3 | 2.15120 | 4.28667 | 7.87017 | 12.91918 | 19.16265 | 26.80251 | 35.59134 |
| 0.4 | * | 2.77400 | 4.62005 | 7.39352 | 10.85356 | 15.09177 | 20.05431 |
| 0.5 | * | 2.19230 | 3.20741 | 4.89938 | 7.05386 | 9.76011 | 12.88982 |
| 0.6 | * | 1.94342 | 2.52131 | 3.60504 | 5.03546 | 6.88171 | 9.02470 |
| 0.7 | * | 1.82742 | 2.18749 | 2.87735 | 3.86149 | 5.17888 | 6.72081 |
| 0.8 | * | 1.76991 | 1.97423 | 2.44917 | 3.13810 | 4.10552 | 5.25154 |
| 0.9 | * | 1.74028 | 1.86097 | 2.19235 | 2.67483 | 3.39898 | 4.26910 |
| 1.0 | * | 1.68490 | 1.79668 | 2.03107 | 2.40771 | 2.91956 | 3.58947 |
| 1.5 | * | 1.62370 | 1.70419 | 1.71427 | 1.83256 | 1.98733 | 2.19094 |
| 2.0 | * | 1.55885 | 1.65207 | 1.67763 | 1.72980 | 1.75725 | 1.85042 |
| 4.0 | * | 1.14059 | 1.54186 | 1.61868 | 1.64360 | 1.65631 | 1.66605 |
| 6.0 | * | 0.81530 | 1.34782 | 1.53641 | 1.59938 | 1.62662 | 1.64115 |
| 10.0 | * | 0.50484 | 0.94960 | 1.26947 | 1.44538 | 1.53212 | 1.57753 |
| 15.0 | * | 0.34108 | 0.66114 | 0.94748 | 1.17735 | 1.33926 | 1.44274 |
| 20.0 | * | 0.25852 | 0.50335 | 0.73604 | 0.94644 | 1.12488 | 1.26527 |

*No frequency can be determined.

Table 3. Frequencies of Single-layered Shell Made of Barite with R = 6 inches and R/h = 25

| $\ell/R$ | $\bar{\lambda}_1$ | $\bar{\lambda}_2$ | $\bar{\lambda}_3$ | $\bar{\lambda}_4$ | $\bar{\lambda}_5$ | $\bar{\lambda}_6$ | $\bar{\lambda}_7$ |
|---|---|---|---|---|---|---|---|
| 0.2 | 12.70109 | 31.36640 | 62.85157 | 94.24792 | 125.70682 | 157.10876 | 188.31525 |
| 0.3 | 5.72400 | 15.86355 | 30.77248 | 51.28117 | 75.96395 | 104.65958 | 125.70137 |
| 0.4 | 3.41902 | 9.02148 | 17.32657 | 28.87850 | 42.86663 | 60.18314 | 79.58576 |
| 0.5 | 2.52165 | 5.90462 | 11.12962 | 18.52941 | 25.18965 | 37.67486 | 50.25341 |
| 0.6 | 2.14077 | 4.26318 | 7.79451 | 10.49503 | 19.08042 | 26.77629 | 35.48634 |
| 0.7 | * | 3.32112 | 5.81420 | 8.93268 | 14.04637 | 19.70364 | 26.08393 |
| 0.8 | * | 2.75048 | 4.55801 | 7.29243 | 10.79252 | 15.12231 | 19.98747 |
| 0.9 | * | 2.40218 | 3.72356 | 5.87402 | 8.57529 | 11.99172 | 14.01778 |
| 1.0 | * | 2.18932 | 3.15052 | 5.85658 | 7.00276 | 9.76337 | 12.62943 |
| 1.5 | * | 1.82855 | 2.06168 | 2.60294 | 3.41726 | 4.25713 | 5.87021 |
| 2.0 | * | 1.60648 | 1.80913 | 2.04933 | 2.39566 | 2.86192 | 3.55582 |
| 4.0 | * | 1.16116 | 1.56103 | 1.63966 | 1.68421 | 1.78310 | 1.87989 |
| 6.0 | * | 0.82350 | 1.36270 | 1.54879 | 1.61155 | 1.64368 | 1.66965 |
| 10.0 | * | 0.50615 | 0.95544 | 1.27791 | 1.45362 | 1.53950 | 1.58477 |
| 15.0 | * | 0.33960 | 0.66260 | 0.95112 | 1.18245 | 1.34488 | 1.44818 |
| 20.0 | * | 0.25917 | 0.50309 | 0.73737 | 0.94899 | 1.12838 | 1.26929 |
| 100.0 | * | 0.05552 | 0.10407 | 0.15398 | 0.20414 | 0.25426 | 0.30420 |
| 200.0 | * | 0.03382 | 0.05550 | 0.07942 | 0.10402 | 0.12890 | 0.15390 |
| 400.0 | * | 0.03882 | 0.03882 | 0.04417 | 0.05550 | 0.06731 | 0.07940 |

*No frequency can be determined.

Table 4. Frequencies of Single-layered Shell Made of Topaz with R = 6 Inches and R/h = 100

| $\ell/R$ | $\bar{\lambda}_1$ | $\bar{\lambda}_2$ | $\bar{\lambda}_3$ | $\bar{\lambda}_4$ | $\bar{\lambda}_5$ | $\bar{\lambda}_6$ | $\bar{\lambda}_7$ |
|---|---|---|---|---|---|---|---|
| 0.2 | 3.77873 | 9.13001 | 17.53133 | 28.90731 | 41.10130 | 60.21945 | 80.11436 |
| 0.3 | 2.44589 | 4.45017 | 7.99372 | 12.97815 | 19.23426 | 26.82705 | 35.64514 |
| 0.4 | 2.17998 | 3.01878 | 4.79536 | 7.49177 | 10.94455 | 15.16146 | 20.11593 |
| 0.5 | * | 2.48815 | 3.44076 | 5.04410 | 7.17418 | 9.83824 | 12.96577 |
| 0.6 | * | 2.26921 | 2.79819 | 3.79788 | 5.19102 | 6.98785 | 9.12037 |
| 0.7 | * | 2.17107 | 2.47124 | 3.11464 | 4.05473 | 5.31692 | 6.84020 |
| 0.8 | * | 2.12274 | 2.34395 | 2.72186 | 3.36809 | 4.27687 | 5.39743 |
| 0.9 | * | 2.09756 | 2.22004 | 2.48752 | 2.93799 | 3.60305 | 4.44285 |
| 1.0 | * | 2.08406 | 2.15619 | 2.34396 | 2.66103 | 3.15403 | 3.79111 |
| 1.5 | * | 2.02143 | 2.07058 | 2.11527 | 2.18244 | 2.30347 | 2.48153 |
| 2.0 | * | 1.97265 | 2.03850 | 2.06510 | 2.09149 | 2.13966 | 2.19560 |
| 4.0 | * | 1.40217 | 1.95978 | 2.01742 | 2.03263 | 2.04040 | 2.05776 |
| 6.0 | * | 0.96518 | 1.72290 | 1.95567 | 2.00473 | 2.02227 | 2.03101 |
| 10.0 | * | 0.58738 | 1.14393 | 1.60791 | 1.86068 | 1.95244 | 1.98932 |
| 15.0 | * | 0.39555 | 0.77677 | 1.14290 | 1.46988 | 1.71620 | 1.85853 |
| 20.0 | * | 0.29999 | 0.58680 | 0.87008 | 1.14239 | 1.39310 | 1.60492 |
| 50.0 | * | 0.13418 | 0.24319 | 0.35703 | 0.47180 | 0.58647 | 0.70052 |
| 100.0 | * | 0.08785 | 0.13420 | 0.18750 | 0.24317 | 0.29984 | 0.35697 |
| 200.0 | * | 0.07174 | 0.08785 | 0.10958 | 0.13421 | 0.16041 | 0.18750 |
| 400.0 | * | 0.07174 | 0.07174 | 0.07886 | 0.08787 | 0.09824 | 0.10959 |

*No frequency can be determined.

Table 5. Frequencies of Single-layered Shell Made of Topaz with R = 6 Inches and R/h = 25

| $\ell/R$ | $\bar{\lambda}_1$ | $\bar{\lambda}_2$ | $\bar{\lambda}_3$ | $\bar{\lambda}_4$ | $\bar{\lambda}_5$ | $\bar{\lambda}_6$ | $\bar{\lambda}_7$ |
|---|---|---|---|---|---|---|---|
| 0.2 | 12.93802 | 31.39377 | 62.84995 | 94.24786 | 125.69214 | 157.10076 | 188.35511 |
| 0.3 | 6.02211 | 15.93219 | 30.95828 | 51.30518 | 76.28224 | 104.68282 | 125.72313 |
| 0.4 | 3.76674 | 9.11568 | 17.48118 | 28.90790 | 43.01371 | 60.20294 | 79.84809 |
| 0.5 | 2.81526 | 6.03685 | 11.28267 | 18.56794 | 25.14958 | 37.68832 | 50.27537 |
| 0.6 | 2.38593 | 4.43809 | 7.95998 | 10.48248 | 19.19910 | 26.81244 | 31.42775 |
| 0.7 | 2.34423 | 3.53863 | 5.99974 | 8.96119 | 14.16324 | 19.74655 | 26.18228 |
| 0.8 | 2.17615 | 3.00670 | 4.76828 | 7.43098 | 10.91740 | 15.17434 | 20.08384 |
| 0.9 | * | 2.68066 | 3.96091 | 6.02043 | 8.70973 | 12.05415 | 15.91328 |
| 1.0 | * | 2.47459 | 3.41540 | 5.02132 | 7.15051 | 9.83703 | 12.59308 |
| 1.5 | * | 2.15030 | 2.34423 | 2.87368 | 3.65121 | 4.24007 | 6.02285 |
| 2.0 | * | 2.01064 | 2.15703 | 2.32354 | 2.64235 | 3.08596 | 3.77514 |
| 4.0 | * | 1.41340 | 1.97450 | 2.03351 | 2.06562 | 2.13030 | 2.19438 |
| 6.0 | * | 0.96779 | 1.73442 | 1.96518 | 2.01385 | 2.03581 | 2.07717 |
| 10.0 | * | 0.58595 | 1.14601 | 1.61366 | 1.86745 | 1.95815 | 1.99480 |
| 15.0 | * | 0.39189 | 0.77603 | 1.14392 | 1.47269 | 1.72046 | 1.86298 |
| 20.0 | * | 0.29465 | 0.58473 | 0.86945 | 1.14288 | 1.39468 | 1.60755 |
| 50.0 | * | 0.12082 | 0.23622 | 0.35246 | 0.46851 | 0.58406 | 0.69862 |
| 100.0 | * | 0.06544 | 0.12079 | 0.17821 | 0.23613 | 0.29422 | 0.35232 |
| 200.0 | * | 0.04130 | 0.06545 | 0.09262 | 0.12078 | 0.14937 | 0.17818 |
| 400.0 | * | 0.03260 | 0.04132 | 0.05273 | 0.06545 | 0.07884 | 0.09261 |

*No frequency can be determined.

Table 6. Frequencies of Single-layered Shell Made of Barite with R = 6 Inches and $\ell/R = 0.5$

| R/h | Present Analysis $\bar{\lambda}_2$ | $\bar{\lambda}_3$ | $\bar{\lambda}_4$ | $\bar{\lambda}_5$ | Classical Analysis* $\bar{\lambda}_2$ | $\bar{\lambda}_3$ | $\bar{\lambda}_4$ | $\bar{\lambda}_5$ |
|---|---|---|---|---|---|---|---|---|
| 20 | 7.26990 | 13.84978 | 23.11686 | 34.24120 | 7.31578 | 14.06013 | 23.14078 | 34.51571 |
| 25 | 5.90462 | 11.12962 | 18.52942 | 27.44840 | 5.93770 | 11.29260 | 18.53970 | 27.63073 |
| 30 | 5.00702 | 9.32109 | 15.48264 | 22.90277 | 5.03338 | 9.45563 | 15.47728 | 23.04409 |
| 50 | 3.28665 | 5.74662 | 9.35641 | 13.81086 | 3.30210 | 5.82842 | 9.38189 | 13.89079 |
| 75 | 2.52131 | 4.02278 | 6.36424 | 9.29048 | 2.52866 | 4.07995 | 6.37714 | 9.34373 |
| 100 | 2.19231 | 3.20741 | 4.89938 | 7.05387 | 2.19448 | 3.25306 | 4.90864 | 7.09425 |
| 150 | 1.92247 | 2.47571 | 3.49467 | 4.86253 | 1.92050 | 2.50025 | 3.50096 | 4.89041 |
| 200 | 1.81776 | 2.16784 | 2.84339 | 3.80848 | 1.81486 | 2.17609 | 2.84841 | 3.83039 |
| 300 | 1.73882 | 1.89111 | 2.26829 | 2.82417 | 1.73547 | 1.91118 | 2.27024 | 2.84057 |
| 400 | 1.71028 | 1.79381 | 2.02891 | 2.41071 | 1.70681 | 1.80932 | 2.02932 | 2.39954 |
| 500 | 1.69689 | 1.74881 | 1.90764 | 2.16106 | 1.69339 | 1.76018 | 1.90754 | 2.16520 |
| 800 | 1.68223 | 1.70027 | 1.76664 | 1.87080 | 1.67871 | 1.70534 | 1.76616 | 1.87863 |

*Longitudinal inertia is not included.

Table 7. Frequencies of Single-layered Shell Made of Barite with R = 6 inches and $\ell/R = 1.0$

| | Present Analysis | | | | Classical Analysis* | | | |
|---|---|---|---|---|---|---|---|---|
| R/h | $\bar{\lambda}_2$ | $\bar{\lambda}_3$ | $\bar{\lambda}_4$ | $\bar{\lambda}_5$ | $\bar{\lambda}_2$ | $\bar{\lambda}_3$ | $\bar{\lambda}_4$ | $\bar{\lambda}_5$ |
| 20 | 2.41132 | 3.74630 | 5.86743 | 8.66363 | 2.44076 | 3.86881 | 6.00673 | 8.77871 |
| 25 | 2.18932 | 3.15052 | 4.85656 | 7.00276 | 2.19449 | 3.25306 | 4.90864 | 7.09403 |
| 30 | 2.05358 | 2.77337 | 4.15742 | 5.90698 | 2.04833 | 2.86362 | 4.19331 | 5.98326 |
| 50 | 1.83073 | 2.16543 | 2.82954 | 3.78217 | 1.81486 | 2.17609 | 2.84842 | 3.83028 |
| 75 | 1.75324 | 1.89283 | 2.26534 | 2.80512 | 1.73547 | 1.91118 | 2.27024 | 2.84050 |
| 100 | 1.68490 | 1.79668 | 2.03108 | 2.40771 | 1.70681 | 1.80932 | 2.02933 | 2.39949 |
| 200 | 1.65697 | 1.70406 | 1.77249 | 1.87169 | 1.67872 | 1.70534 | 1.76617 | 1.87861 |
| 300 | 1.65107 | 1.68701 | 1.72119 | 1.76164 | 1.67346 | 1.68538 | 1.71300 | 1.76536 |
| 400 | 1.64846 | 1.67020 | 1.68496 | 1.72208 | 1.67162 | 1.67834 | 1.69399 | 1.72397 |
| 500 | 1.64688 | 1.66747 | 1.68130 | 1.70360 | 1.67076 | 1.67507 | 1.68513 | 1.70447 |
| 600 | 1.64579 | 1.66595 | 1.67860 | 1.69967 | 1.67030 | 1.67329 | 1.68029 | 1.69379 |
| 800 | 1.64432 | 1.66439 | 1.67539 | 1.68780 | 1.66984 | 1.67152 | 1.67547 | 1.68309 |

*Longitudinal inertia is not included.

Table 8. Frequencies of Single-layered Shell Made of Topaz with R = 6 Inches and $\ell/R = 0.5$

| R/h | Present Analysis | | | | Classical Analysis* | | | |
|---|---|---|---|---|---|---|---|---|
| | $\bar{\lambda}_2$ | $\bar{\lambda}_3$ | $\bar{\lambda}_4$ | $\bar{\lambda}_5$ | $\bar{\lambda}_2$ | $\bar{\lambda}_3$ | $\bar{\lambda}_4$ | $\bar{\lambda}_5$ |
| 20 | 7.38661 | 14.01616 | 23.15357 | 34.40271 | 7.41058 | 14.10969 | 23.17093 | 34.53593 |
| 25 | 6.03685 | 11.28267 | 18.56794 | 27.57168 | 6.05412 | 11.35425 | 18.57731 | 27.65598 |
| 30 | 5.15652 | 9.47037 | 15.52061 | 23.01032 | 5.17020 | 9.52917 | 15.52230 | 23.07436 |
| 50 | 3.49937 | 5.91131 | 9.44297 | 13.90555 | 3.50712 | 5.94698 | 9.45600 | 13.94095 |
| 75 | 2.78582 | 4.22249 | 6.47897 | 9.39466 | 2.79108 | 4.24759 | 6.48567 | 9.41814 |
| 100 | 2.48815 | 3.44076 | 5.04410 | 7.17417 | 2.49234 | 3.46098 | 5.04884 | 7.19196 |
| 150 | 2.25340 | 2.74941 | 3.69191 | 5.01883 | 2.25484 | 2.76537 | 3.69496 | 5.03112 |
| 200 | 2.16597 | 2.46185 | 3.08145 | 3.99881 | 2.16558 | 2.47616 | 3.08374 | 2.00848 |
| 300 | 2.10107 | 2.25347 | 2.55769 | 3.06924 | 2.09949 | 2.24691 | 2.55929 | 3.07650 |
| 400 | 2.07783 | 2.15604 | 2.34395 | 2.66843 | 2.07586 | 2.16093 | 2.34822 | 2.67465 |
| 500 | 2.06697 | 2.11414 | 2.24337 | 2.46089 | 2.06484 | 2.11996 | 2.24382 | 2.46659 |
| 800 | 2.05513 | 2.07144 | 2.12533 | 2.21982 | 2.05282 | 2.07465 | 2.12493 | 2.21929 |

*Longitudinal inertia is not included.

Table 9. Frequencies of Single-layered Shell Made of Topaz with R = 6 Inches and $\ell/R = 1.0$

| | Present Analysis | | | | Classical Analysis* | | | |
|---|---|---|---|---|---|---|---|---|
| R/h | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ | $\lambda_5$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ | $\lambda_5$ |
| 20 | 2.69039 | 3.99080 | 6.02394 | 8.80548 | 2.71170 | 4.04520 | 6.12183 | 8.85787 |
| 25 | 2.47459 | 3.41540 | 5.02131 | 7.15051 | 2.49235 | 3.46098 | 5.04884 | 7.19175 |
| 30 | 2.35694 | 3.05755 | 4.33799 | 6.06446 | 2.36467 | 3.09779 | 4.35659 | 6.09881 |
| 50 | 2.16749 | 2.44405 | 3.07425 | 3.98676 | 2.16557 | 2.47616 | 3.08375 | 4.00838 |
| 75 | 2.10616 | 2.25201 | 2.55277 | 3.06045 | 2.09949 | 2.24691 | 2.55929 | 3.07644 |
| 100 | 2.08407 | 2.15619 | 2.34390 | 2.66103 | 2.07586 | 2.16093 | 2.34823 | 2.67461 |
| 200 | 2.04177 | 2.07307 | 2.12680 | 2.21944 | 2.05283 | 2.07465 | 2.12493 | 2.21927 |
| 300 | 2.03721 | 2.05888 | 2.08430 | 2.12241 | 2.04853 | 2.05828 | 2.08095 | 2.12427 |
| 400 | 2.03536 | 2.05401 | 2.06972 | 2.08888 | 2.04702 | 2.05252 | 2.06534 | 2.08999 |
| 500 | 2.03430 | 2.05176 | 2.06355 | 2.07347 | 2.04633 | 2.04985 | 2.05807 | 2.07394 |
| 600 | 2.03359 | 2.04496 | 2.05226 | 2.06514 | 2.04505 | 2.04839 | 2.05411 | 2.06517 |
| 800 | 2.03266 | 2.04373 | 2.05009 | 2.05690 | 2.04557 | 2.04695 | 2.05017 | 2.05640 |

*Longitudinal inertia is not included.

Table 10. Frequencies of Single-layered Shell Made of Barite with R = 6 Inches and $\ell/R$ = 10.0

| R/h | Present Analysis | | | | Classical Analysis* | | | |
|---|---|---|---|---|---|---|---|---|
| | $\bar{\lambda}_2$ | $\bar{\lambda}_3$ | $\bar{\lambda}_4$ | $\bar{\lambda}_5$ | $\bar{\lambda}_2$ | $\bar{\lambda}_3$ | $\bar{\lambda}_4$ | $\bar{\lambda}_5$ |
| 20 | 0.50678 | 0.95695 | 1.27998 | 1.45571 | 1.66935 | 1.66962 | 1.67025 | 1.67147 |
| 25 | 0.50615 | 0.95544 | 1.27791 | 1.45362 | 1.66931 | 1.66948 | 1.66989 | 1.67067 |
| 30 | 0.50572 | 0.95434 | 1.27640 | 1.45212 | 1.66929 | 1.66941 | 1.66969 | 1.67024 |
| 50 | 0.50489 | 0.95184 | 1.27288 | 1.44869 | 1.66927 | 1.66931 | 1.66941 | 1.66961 |
| 75 | 0.50469 | 0.95037 | 1.27072 | 1.44659 | 1.66926 | 1.66928 | 1.66932 | 1.66941 |
| 100 | 0.50484 | 0.94960 | 1.26947 | 1.44537 | 1.66925 | 1.66927 | 1.66929 | 1.66934 |
| 200 | 0.50663 | 0.94866 | 1.26729 | 1.44313 | 1.66925 | 1.66925 | 1.66926 | 1.66927 |

*Longitudinal inertia is not included.

Table 11. Frequencies of Single-layered Shell Made of Topaz with R = 6 Inches and $\ell/R = 10.0$

| | Present Analysis | | | | Classical Analysis* | | | |
|---|---|---|---|---|---|---|---|---|
| R/h | $\bar{\lambda}_2$ | $\bar{\lambda}_3$ | $\bar{\lambda}_4$ | $\bar{\lambda}_5$ | $\bar{\lambda}_2$ | $\bar{\lambda}_3$ | $\bar{\lambda}_4$ | $\bar{\lambda}_5$ |
| 20 | 0.58609 | 1.14669 | 1.61509 | 1.86917 | 2.04519 | 2.04539 | 2.04591 | 2.04691 |
| 25 | 0.58595 | 1.14601 | 1.61366 | 1.86745 | 2.04514 | 2.04528 | 2.04561 | 2.04625 |
| 30 | 0.58589 | 1.14552 | 1.61261 | 1.86622 | 2.04513 | 2.04522 | 2.04545 | 2.04590 |
| 50 | 0.58605 | 1.14450 | 1.61018 | 1.86340 | 2.04510 | 2.04514 | 2.04522 | 2.04538 |
| 75 | 0.58663 | 1.14404 | 1.60872 | 1.86168 | 2.04510 | 2.04511 | 2.04515 | 2.04522 |
| 100 | 0.58738 | 1.14393 | 1.60791 | 1.86069 | 2.04509 | 2.04510 | 2.04512 | 2.04516 |
| 200 | 0.59106 | 1.14460 | 1.60668 | 1.85888 | 2.04509 | 2.04509 | 2.04510 | 2.04511 |

*Longitudinal inertia is not included.

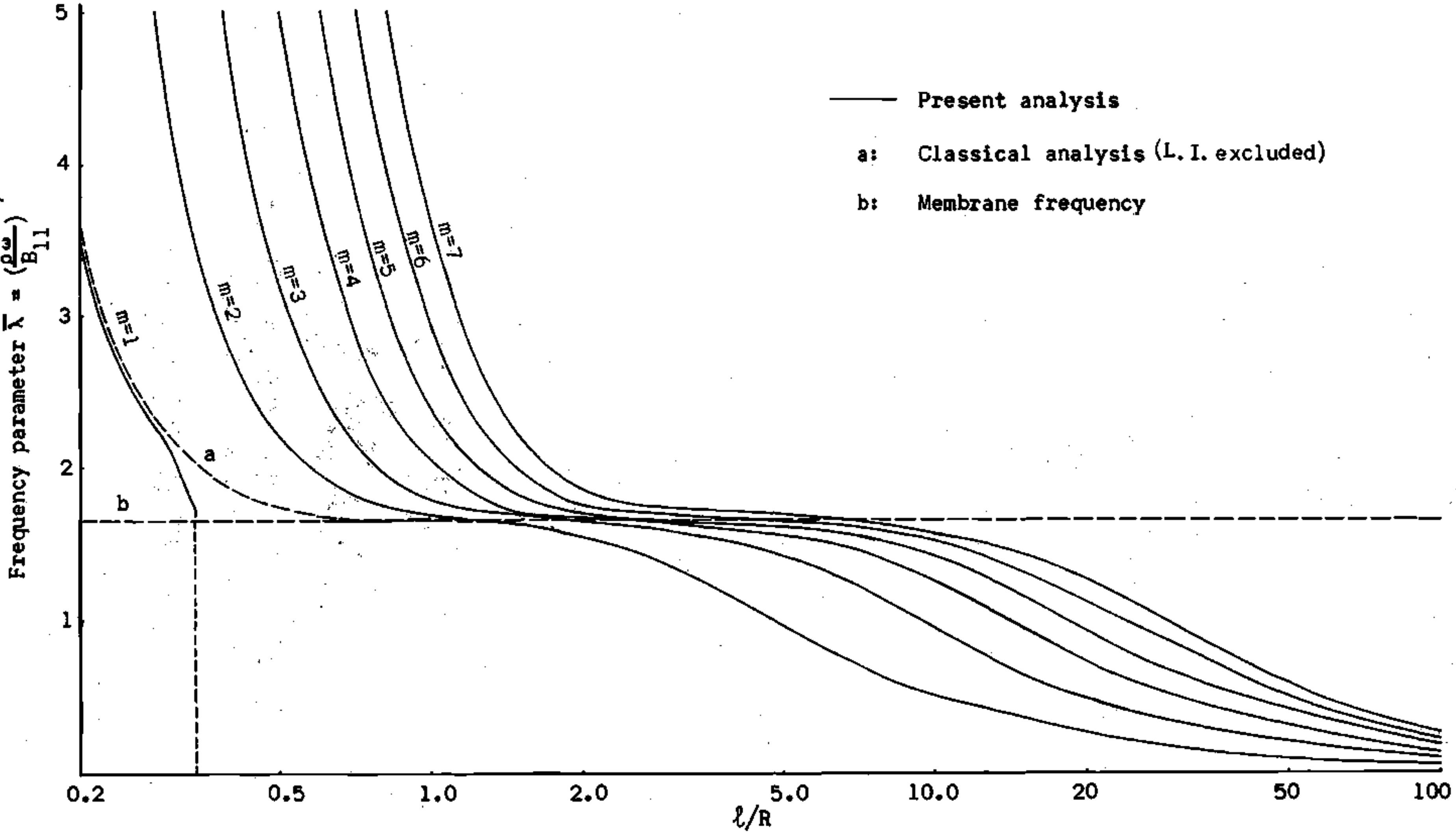



Figure 2. Frequencies of Single-layered Shell Made of Barite with R/h = 100 and R = 6 Inches.

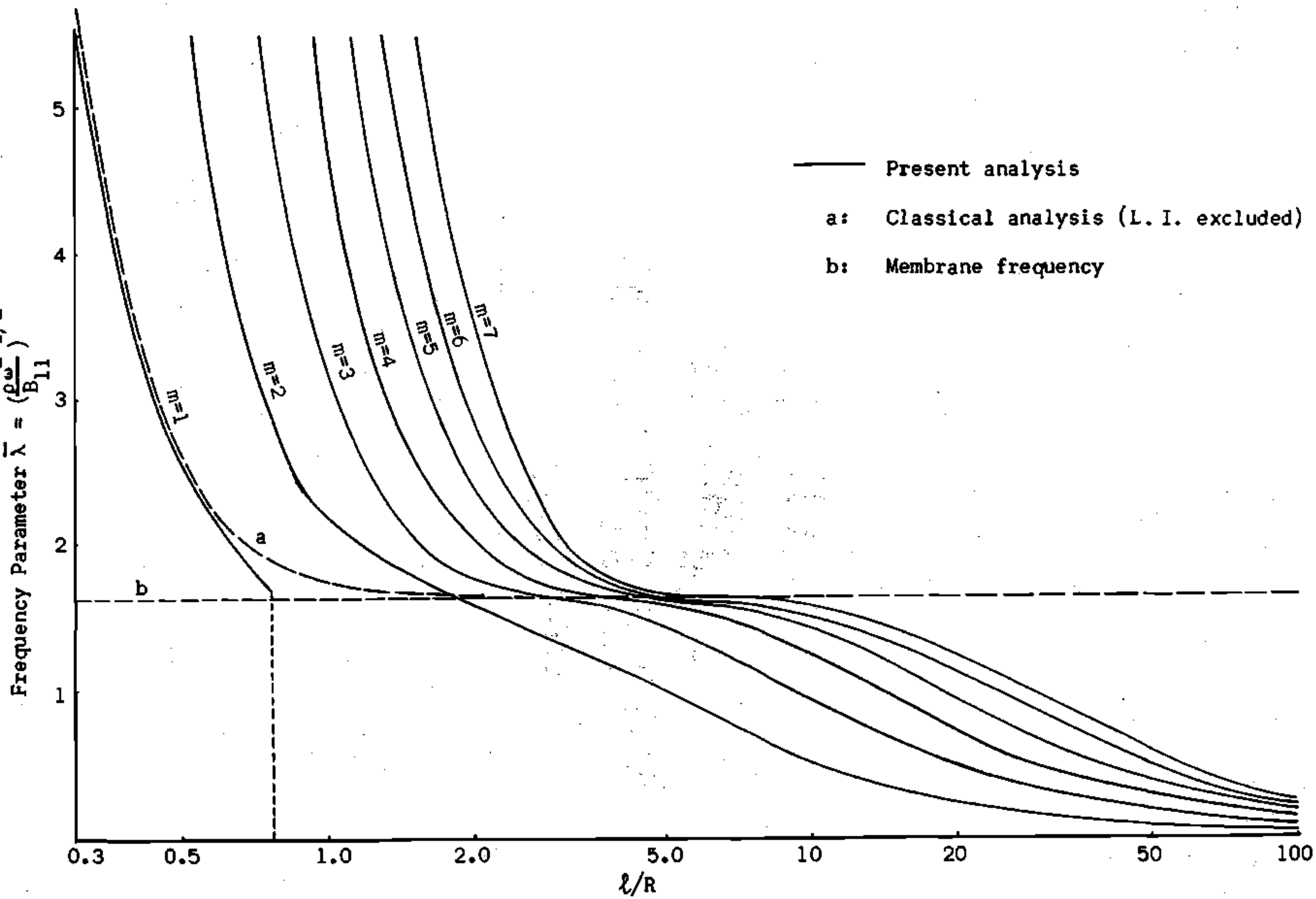



Figure 3. Frequencies of Single-layered Shell Made of Barite with R/h = 25 and R = 6 Inches.

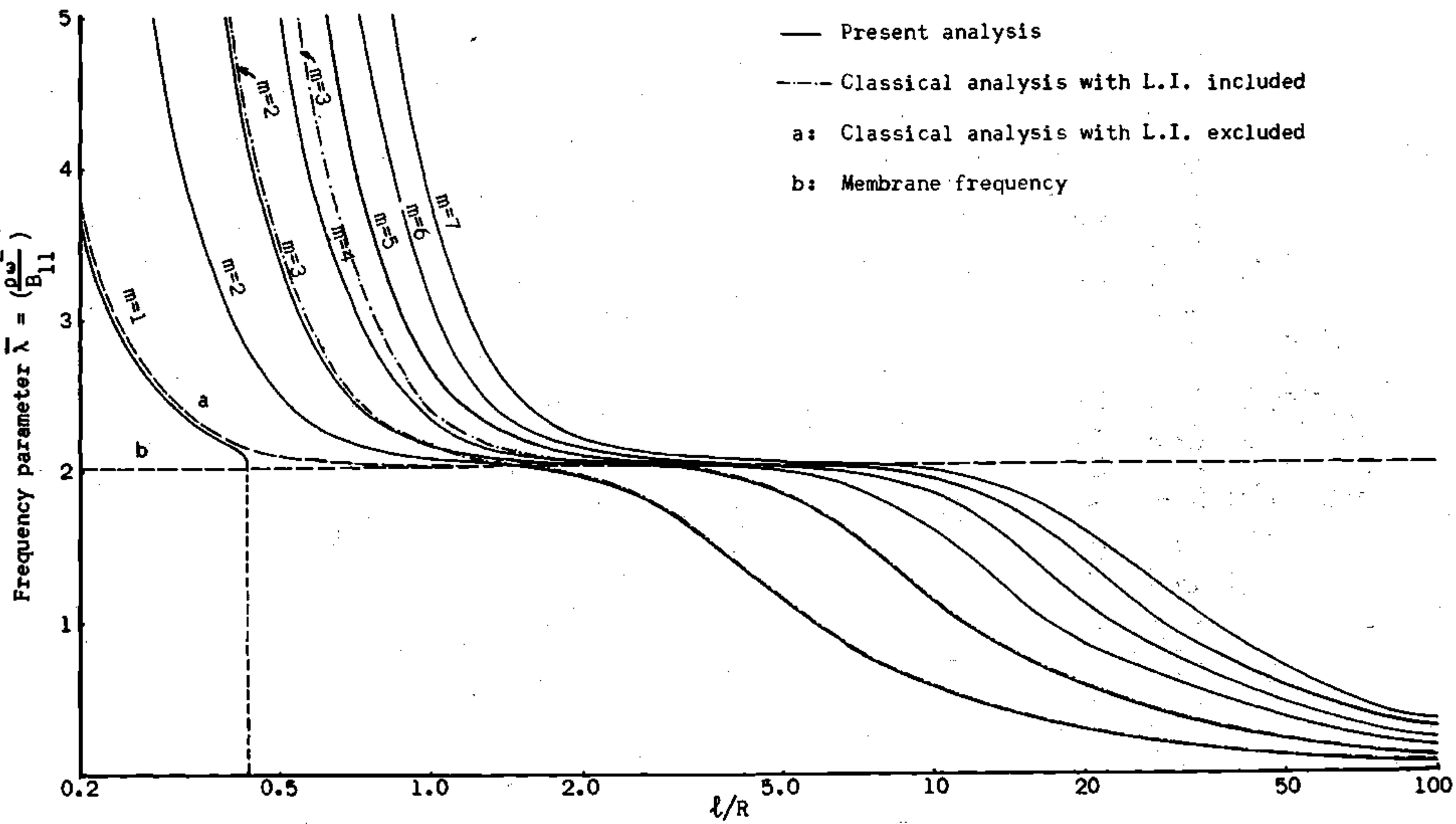



Figure 4. Frequencies of Single-layered Shell Made of Topaz with R/h = 100 and R = 6 Inches.

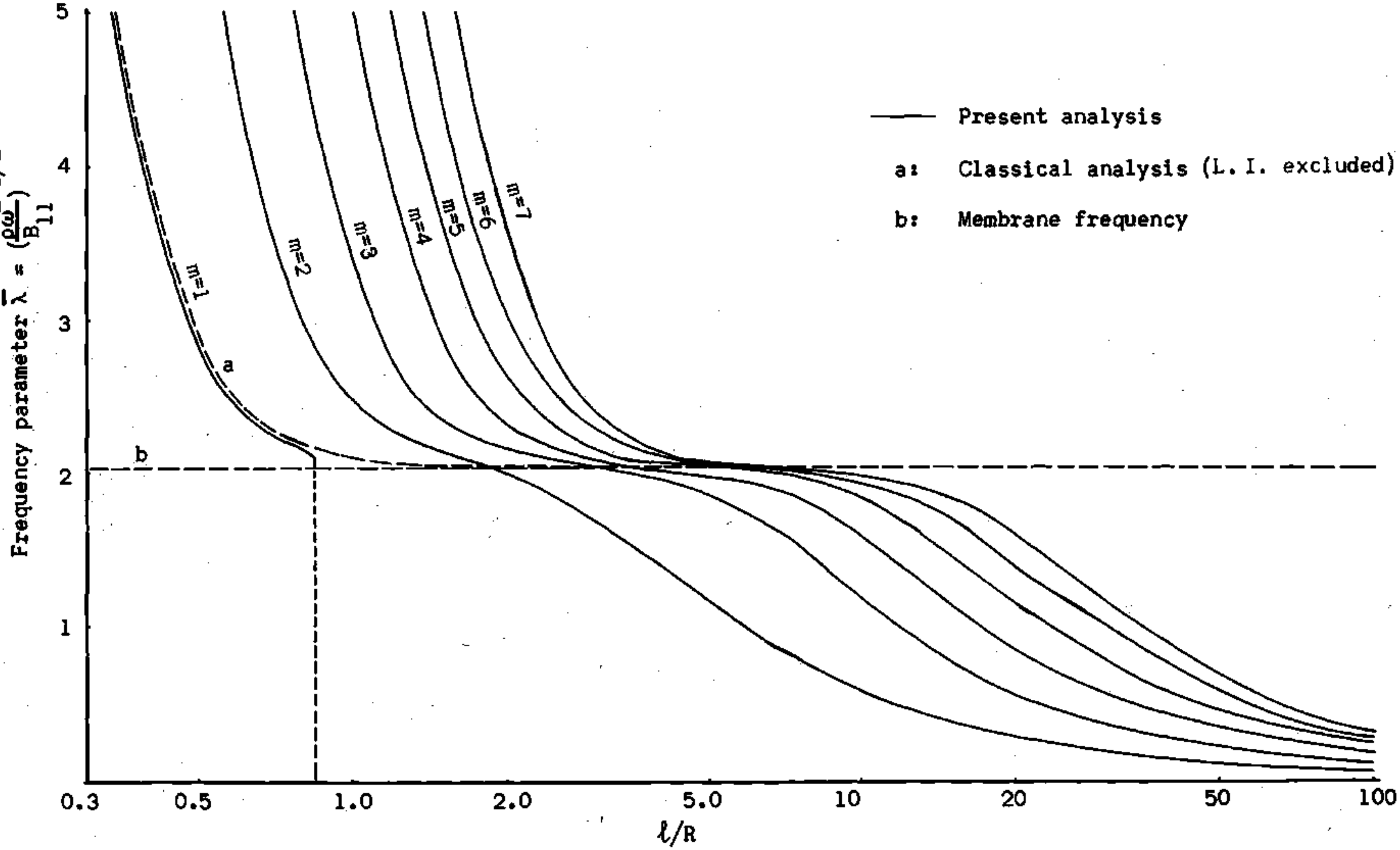



Figure 5. Frequencies of Single-layered Shell Made of Topaz with R/h = 25 and R = 6 Inches.

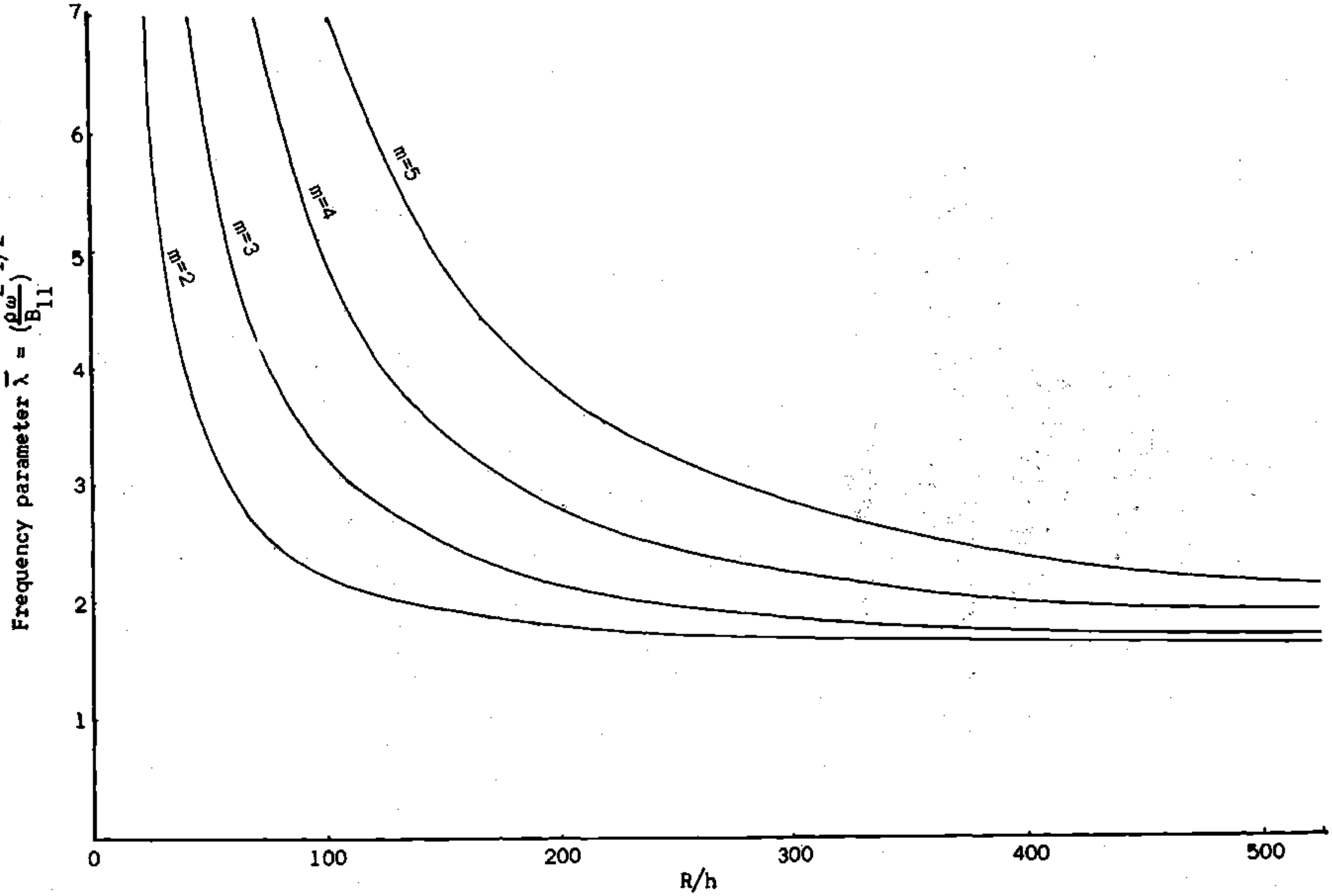



Figure 6. Frequencies of Single-layered Shell Made of Barite with $\ell/R = 0.5$ and $R = 6$ Inches.

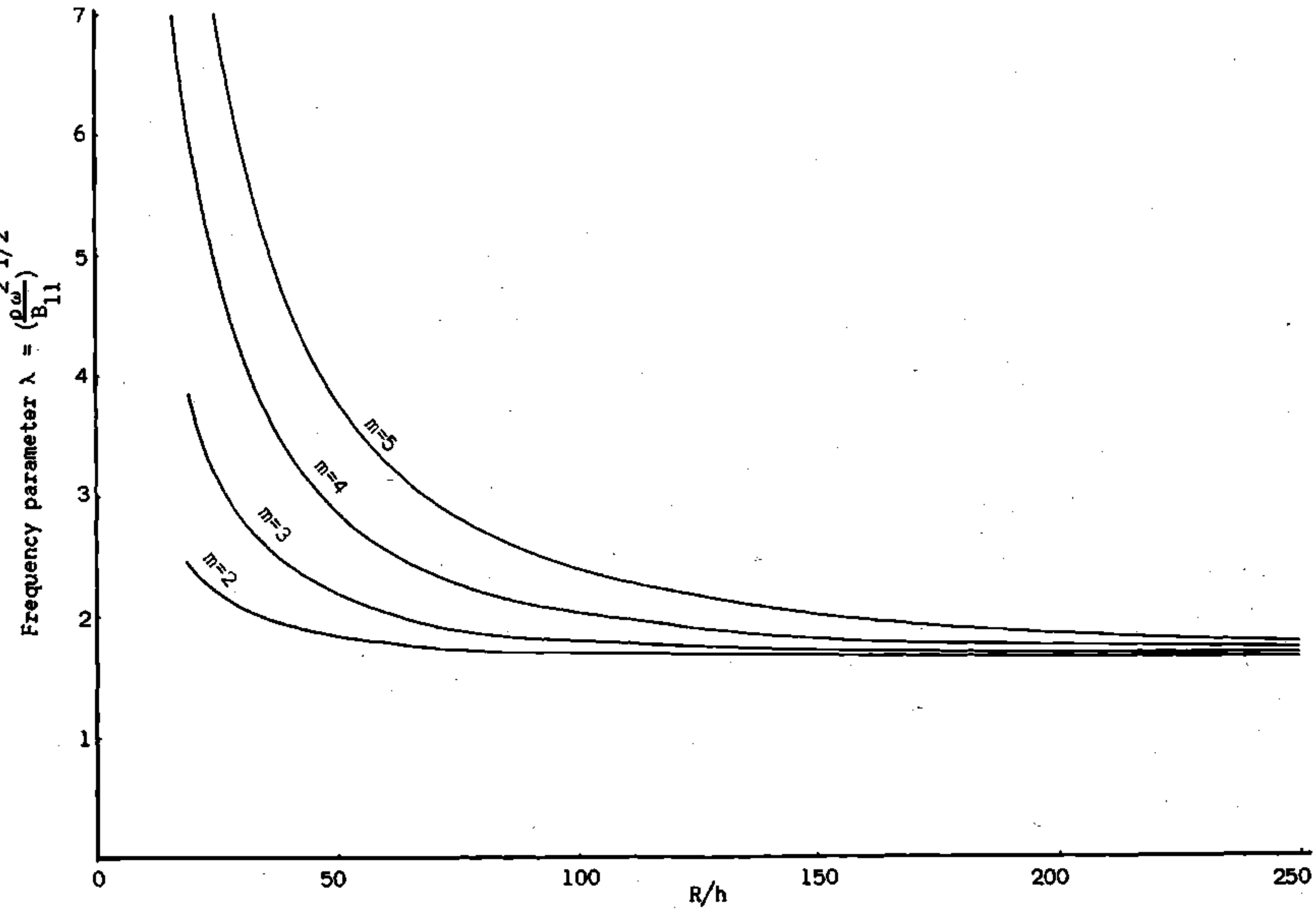



Figure 7. Frequencies of Single-layered Shell Made of Barite with $\ell/R = 1.0$ and R = 6 Inches.

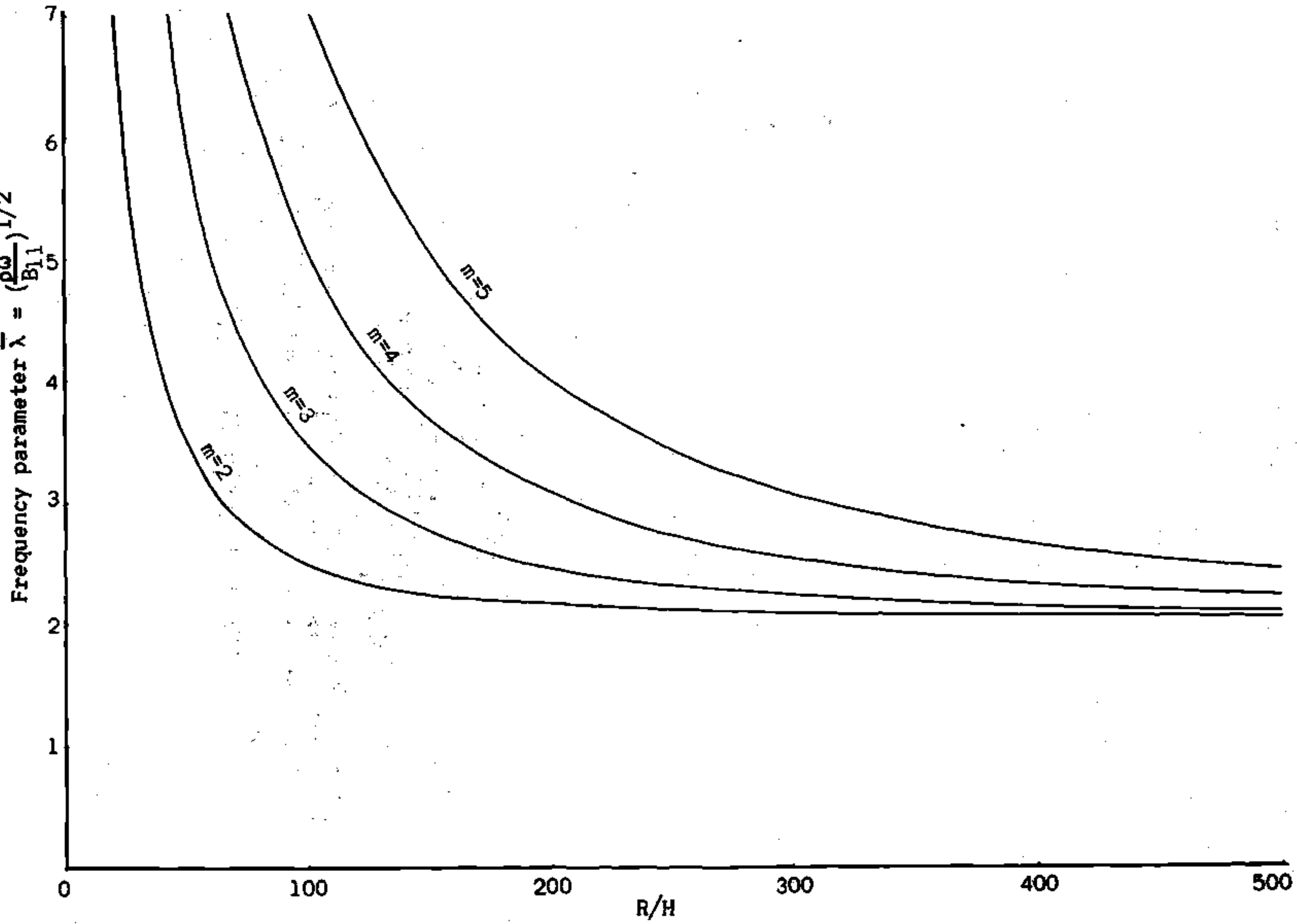



Figure 8. Frequencies of Single-layered Shell Made of Topaz with $\ell/R = 0.5$ and $R = 6$ Inches.

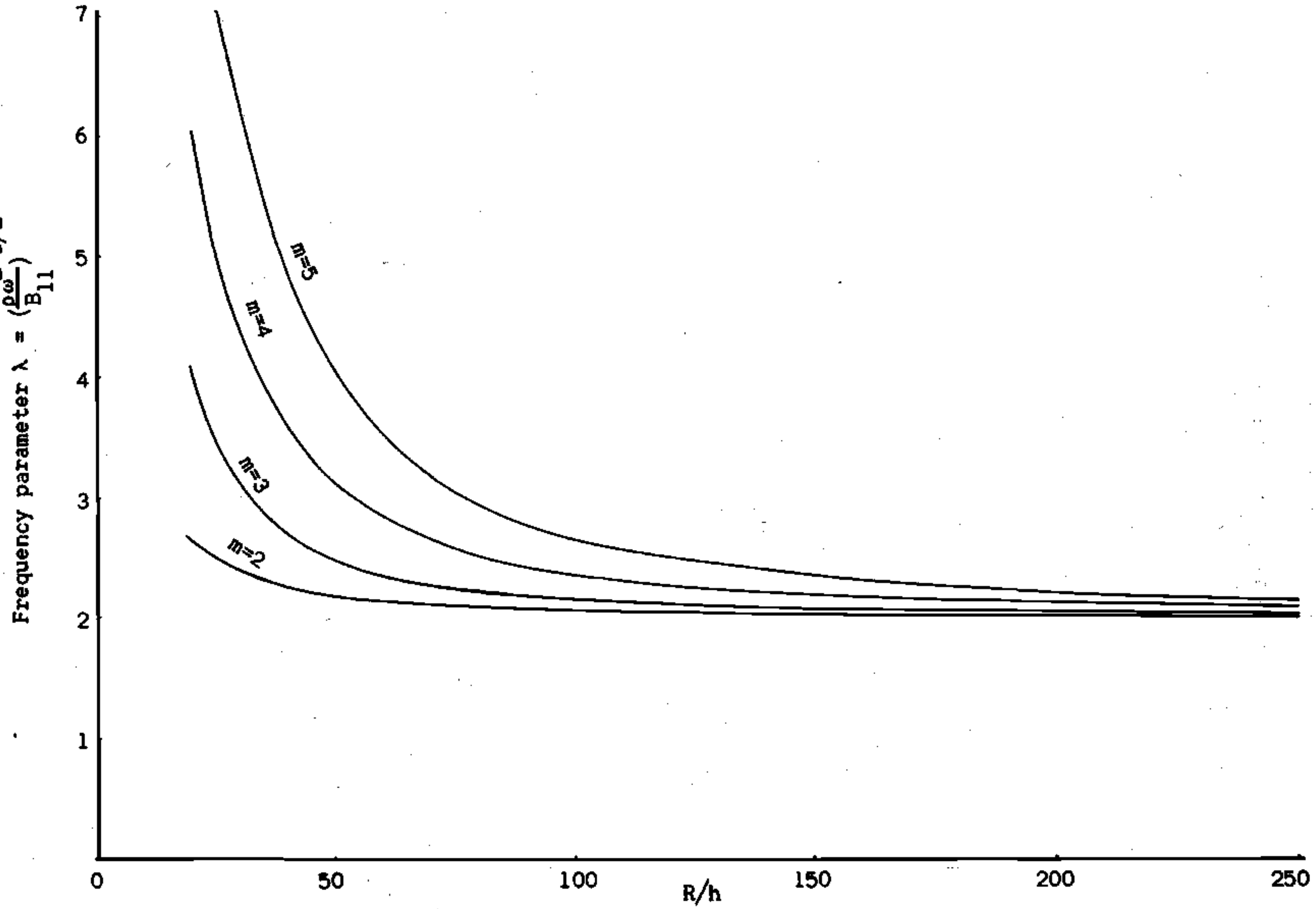



Figure 9. Frequencies of Single-layered Shell Made of Topaz with $\ell/R = 1.0$ and $R = 6$ Inches.

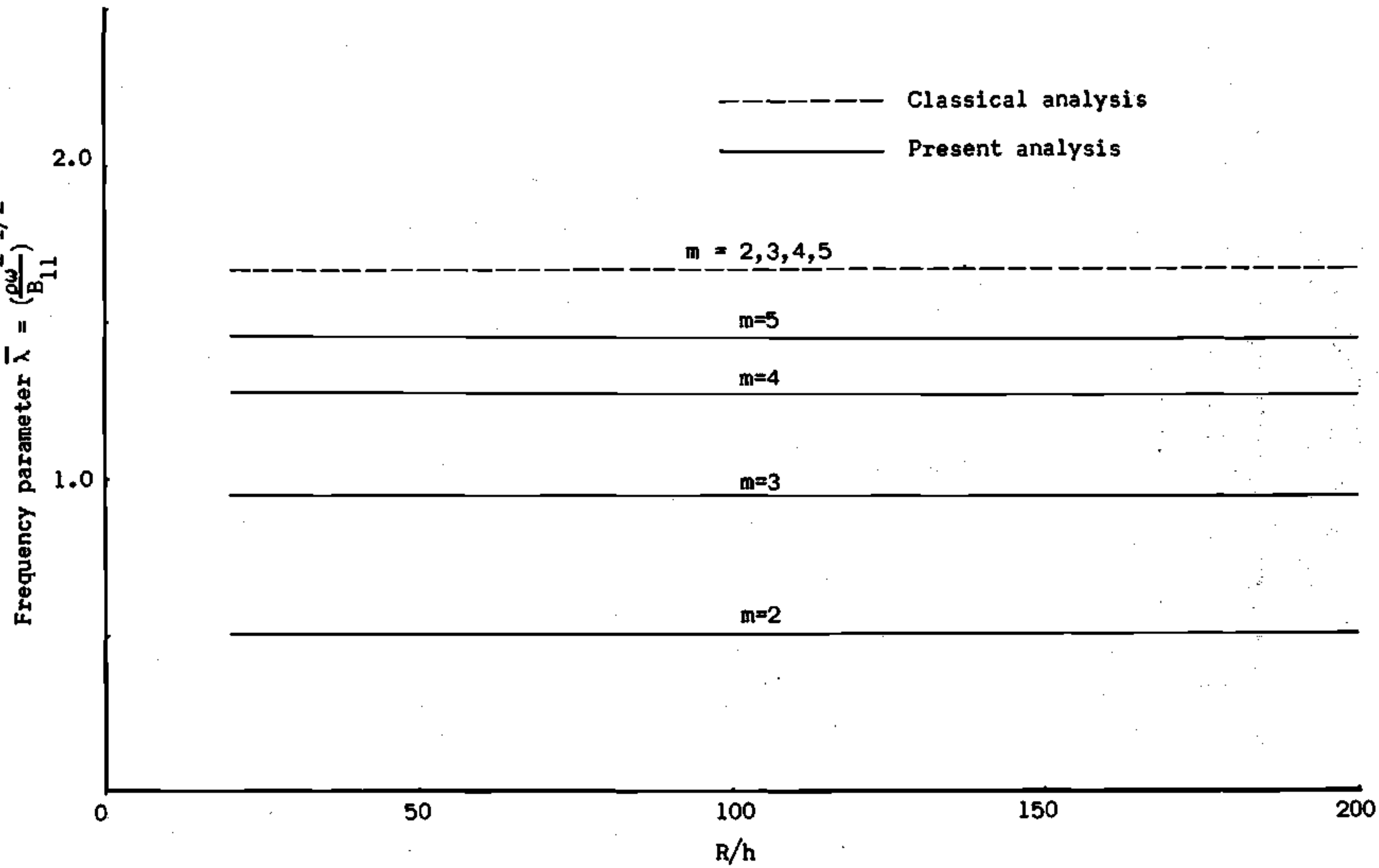



Figure 10. Frequencies of Single-layered Shell Made of Barite with $\ell/R = 10.0$ and R = 6 Inches.

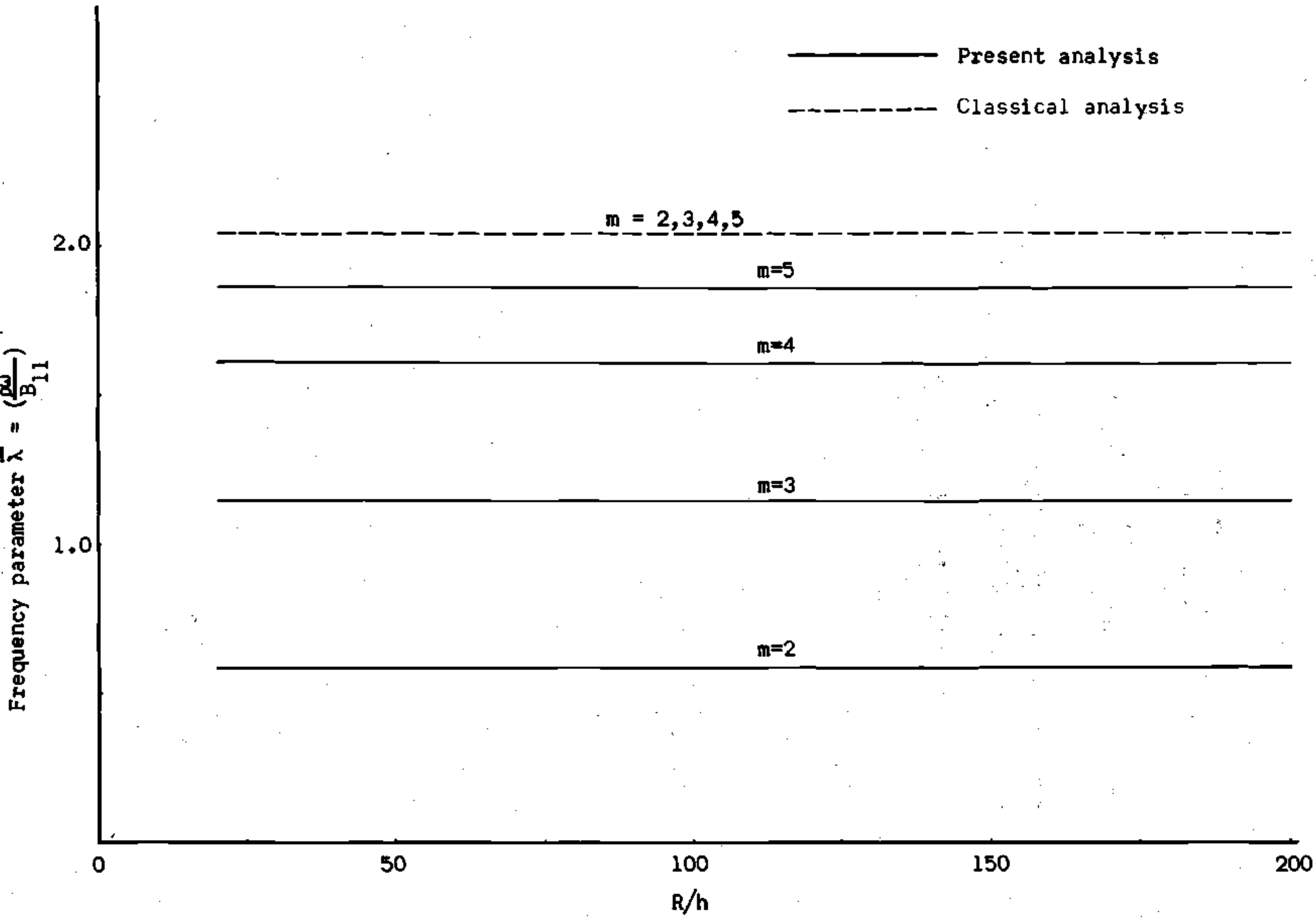
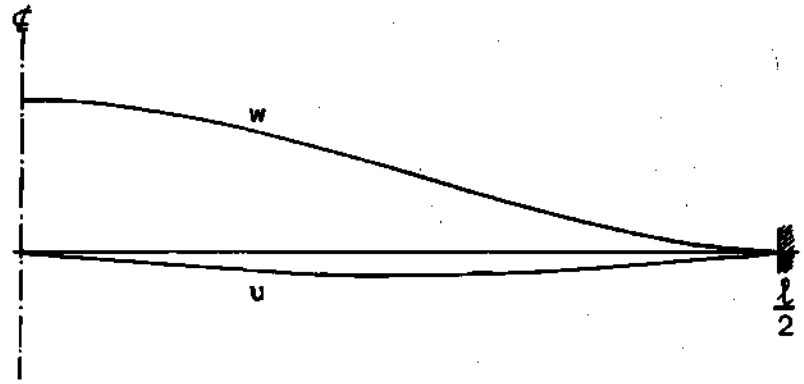
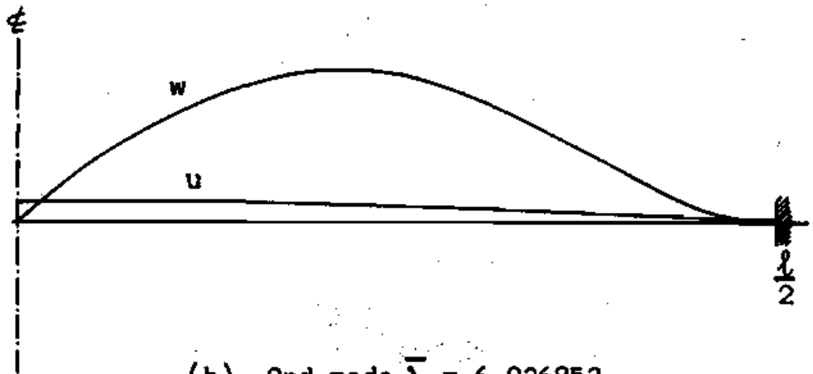


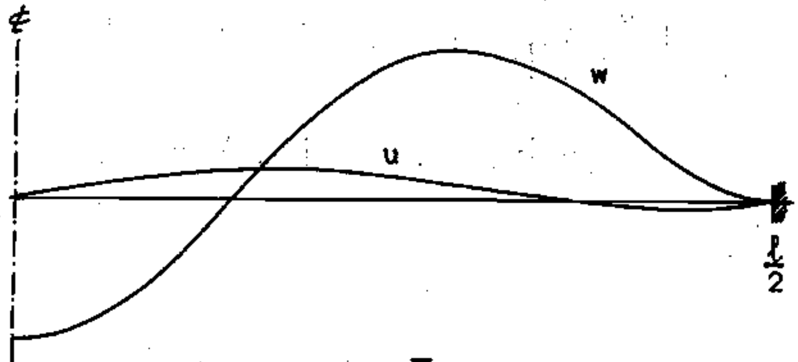

Figure 11. Frequencies of Single-layered Shell Made of Topaz with $\ell/R = 10.0$ and R = 6 Inches.




(a) 1st mode $\overline{\lambda}$ = 2.815262




(b) 2nd mode $\overline{\lambda}$ = 6.036853




(c) 3rd mode $\overline{\lambda}$ = 11.282673

Figure 12. Mode Shapes of Single-layered Shell Made of Topaz with $R/h = 25$, $R/\ell = 2.0$ and $R = 6$ Inches.

single-layered shell. For the purpose of checking the derivation as well as the computer programming, the fundamental lower branch frequency of a two-layered shell having layers of the same isotropic material are calculated according to the multi-layered shell analysis and the results are compared to the values calculated according to the single-layered shell for which detailed discussions were presented in the previous section. Again an iterative procedure is used to determine the natural frequencies from the frequency equation, (5.21), for the multi-layered shell. To give some idea of the variation of the value of the determinant with change in frequency parameter in the process of frequency determination, two cases of double-layered shells of isotropic material are presented (Figure 13). As can be seen in the figure, the variation of the determinant versus frequency parameter is quite complicated when contrasted with a single-layered shell.

The fundamental frequency parameter, calculated for various thickness combinations while maintaining total thickness constant, is tabulated in Table 12. Although the mean radius and the total thickness are the same for all four cases, there are slight differences in the results. The maximum difference is about 4 per cent.

Numerical results for a finite length, fixed-fixed supported two-layered cylindrical shell whose inner layer (the number one layer) is made of barite and outer layer (the number two layer) of topaz are obtained. In equation (5.11), if only even numbered values of n are taken, one obtains vibration symmetrical about the middle section of the cylindrical shell. Similarly, if n is taken to be odd only, the antisymmetrical modes of vibration result. The convergence of the

Fourier series is rapid in the cases considered here. The coupled frequency obtained by using the first five terms of the Fourier series of each loading function differs from the one obtained by using the first six terms by less than 0.1 percent.

For multi-layered shells, the coupled frequency parameter $\bar{\lambda}$ may be expressed in terms of the density $\rho$ and the elasticity constant $B_{11}$ of the material of any layer. For convenience of presentation, one designates $\bar{\lambda}^j = \left(\frac{\rho^j \omega^2}{B_{11}^j}\right)^{1/2}$ as frequency parameter expressed in terms of $\rho^j$ and $B_{11}^j$ of the jth layer material. Two coupled frequencies related to $\bar{\lambda}^{(1)}$ and $\bar{\lambda}^{(2)}$ for $R_1$ = 6 inches and $\ell/R_1$ = 1.0 are given in Table 13.

For $R_1$ = 6 inches and $R_1/h_1 = R_1/h_2 = 100$, four frequencies for different length shells are given in Table 14. It is clear that when the ratio $\ell/R_1$ increases the frequency decreases. Three mode shapes for $\ell/R_1$ = 1.0 are given in Figure 14.

The method presented in this analysis is very flexible. Although the illustrative example given in the first section is for a case of fixed-fixed supported edges and is only one of the particular forms of the solution of equation (3.25), other cases, which are indicated in Appendix B, may be treated similarly. The method can also be easily extended to cylindrical shells with any other boundary conditions. Although the numerical examples given here are for one-layered and two-layered cylindrical shells made of barite and topaz, shells consisting of a large number of layers and/or made of other orthotropic materials, can be investigated, using the same procedures, without difficulty.

Table 12. Fundamental Frequency of Two-layered Isotropic Cylindrical Shell with $\ell$ = 6 Inches and $h_1 + h_2$ = 0.096 Inches

| No. of layers | One | Two | Two | Two |
|---|---|---|---|---|
| Geometric dimensions (in.) | $h_1 = 0.096$<br>$R = 6.0$ | $h_1 = 0.048$<br>$h_2 = 0.048$<br>$R_1 = 5.976$<br>$R_2 = 6.024$ | $h_1 = 0.060$<br>$h_2 = 0.036$<br>$R_1 = 5.982$<br>$R_2 = 6.030$ | $h_1 = 0.036$<br>$h_2 = 0.060$<br>$R_1 = 5.970$<br>$R_2 = 6.018$ |
| Frequency parameter $\bar{\lambda} = \left(\frac{\rho\omega^2}{B_{11}}\right)^{1/2}$ | 1.9015 | 1.8282 | 1.8256 | 1.8195 |

Table 13. Couple Frequencies of Two-layered Cylindrical Shell Whose Inner Layer is Made of Barite and Outer Layer of Topaz with $R_1 = 6$ Inches and $\ell/R_1 = 1.0$

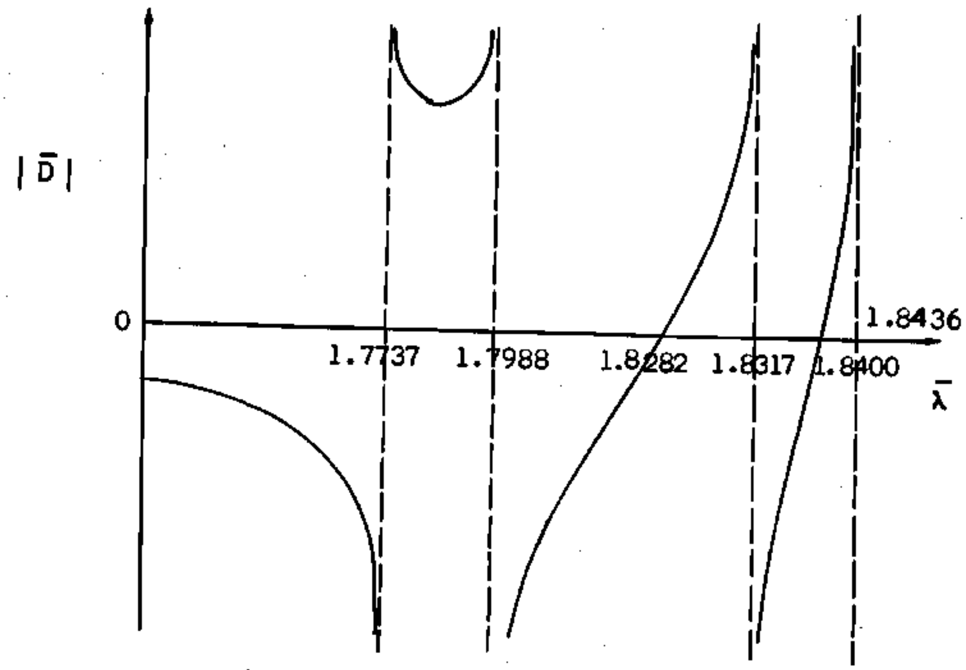
| $h_1$ ($10^{-2}$ in.) | $h_2$ ($10^{-2}$ in.) | $\bar{\lambda}_2^{(1)}$ | $\bar{\lambda}_3^{(1)}$ | $\bar{\lambda}_2^{(2)}$ | $\bar{\lambda}_3^{(2)}$ |
|---|---|---|---|---|---|
| 6.0 | 7.2 | 1.72506 | 1.79673 | 0.86684 | 0.90286 |
| 7.2 | 6.0 | 1.74113 | 1.85117 | 0.87492 | 0.93022 |
| 6.0 | 6.0 | 1.72504 | 1.79673 | 0.86683 | 0.90286 |
| 4.8 | 6.0 | 1.71169 | 1.75218 | 0.86012 | 0.88047 |
| 6.0 | 4.8 | 1.72502 | 1.79672 | 0.86682 | 0.90285 |
| 3.6 | 6.0 | 1.69788 | 1.71761 | 0.85318 | 0.86310 |
| 6.0 | 3.6 | 1.72501 | 1.79671 | 0.86682 | 0.90285 |

$R_2 = R_1 + \frac{1}{2}(h_1 + h_2)$

Table 14. Coupled Frequencies of Two-layered Cylindrical Shell Whose Inner Layer is Made of Barite and Outer Layer of Topaz with $R_1 = 6$ Inches and $R_1/h_1 = R_1/h_2 = 100$

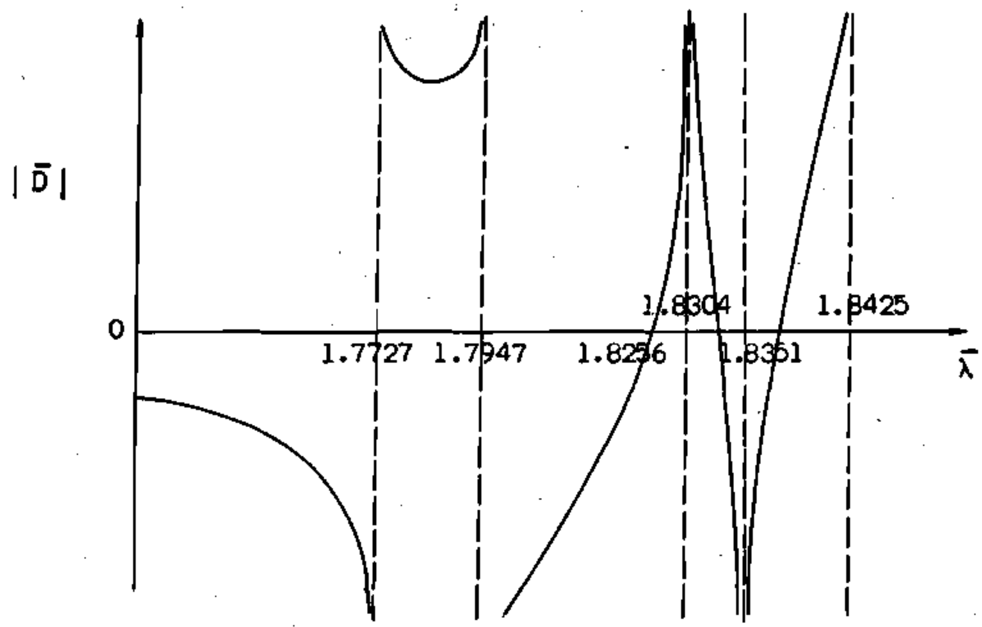
| $\ell/R_1$ | $\bar{\lambda}_2^{(1)*}$ | $\bar{\lambda}_3^{(1)}$ | $\bar{\lambda}_4^{(1)}$ | $\bar{\lambda}_5^{(1)}$ |
|---|---|---|---|---|
| 1.0 | 1.72492 | 1.79682 | 2.03114 | 2.06290 |
| 2.0 | 1.55882 | 1.65208 | 1.69981 | 1.72983 |
| 5.0 | 0.95532 | 1.45330 | 1.66065 | 1.71507 |
| 10.0 | 0.50266 | 0.94885 | 1.44531 | 1.53210 |

$$* \quad \bar{\lambda}^{(1)} = \frac{\rho^{(1)}\omega^2}{B_{11}^{(1)}}$$




(a) $h_1 = 0.048$ in., $R_1 = 5.976$ in.

$h_2 = 0.048$ in., $R_2 = 6.024$ in.




(b) $h_1 = 0.060$ in., $R_1 = 5.982$ in.

$h_2 = 0.036$ in., $R_2 = 6.030$ in.

Figure 13. Frequencies Versus Determinant Values for a Two-layered Isotropic Cylindrical Shell with $\ell = 6$ Inches and $h_1 + h_2 = 0.096$ Inches.

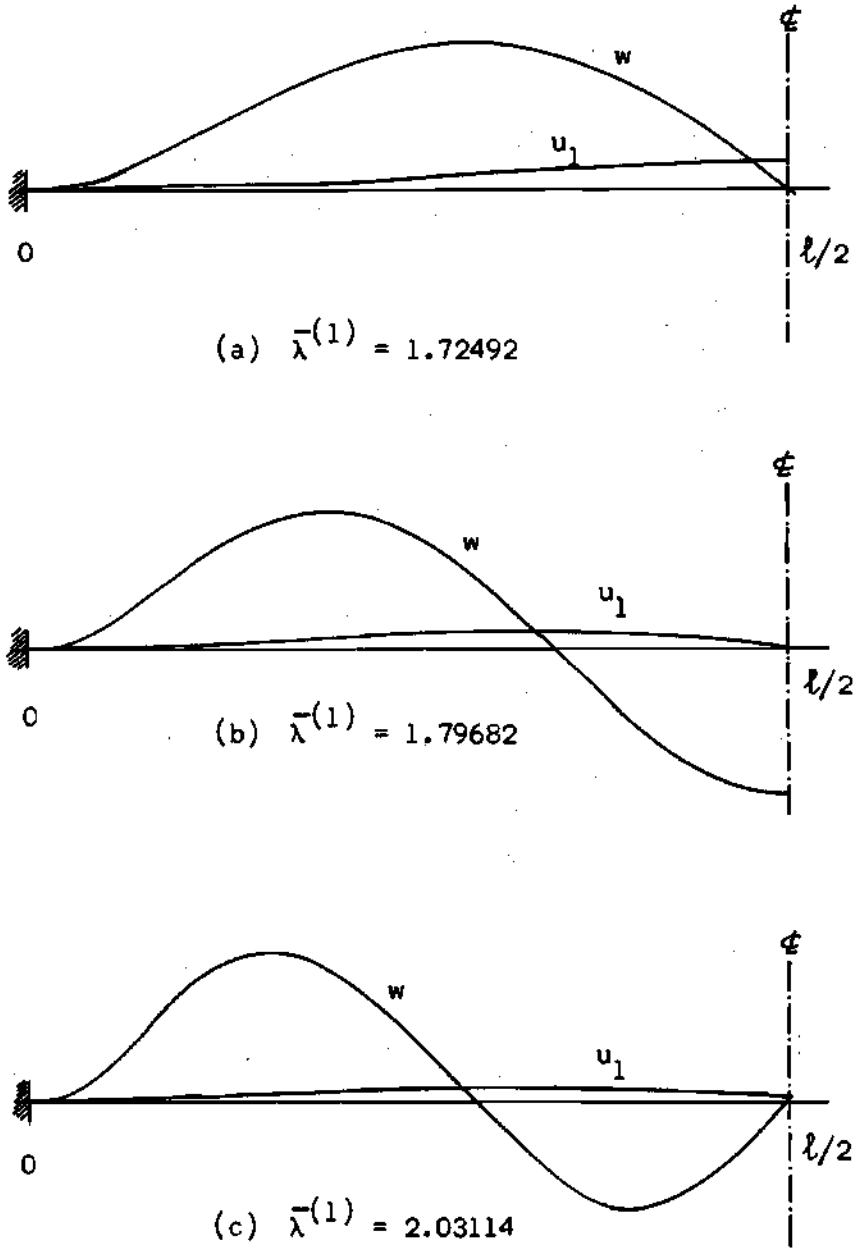



Figure 14. Mode Shapes of a Two-layered Shell Whose Inner Layer is Made of Barite and Outer Layer of Topaz with $R_1/h_1 = R_1/h_2 = 100$, $\ell/R_1 = 1.0$ and $R_1 = 6$ Inches.

# CHAPTER VI

## CONCLUSIONS

This analysis presents a closed solution for the rotationally symmetric vibration of layered cylindrical shells governed by a set of partial differential equations which have been derived without using the hypothesis of nondeformable normals. The problem of determining the coupled frequencies of an N-layered, homogeneous orthotropic cylindrical shell is posed in terms of a set of inhomogeneous sixth order linear differential equations with the unknown tractions between the contact surfaces as external loads. These tractions couple the differential equations. Both the partial differential equations which characterize the dynamic behavior, and the boundary conditions are satisfied exactly. The resulting transcendental characteristic equations (or frequency equations) are solved numerically. As indicated in Chapter V, the convergence of the Fourier series is rapid.

The form of the roots of equation (3.25) must be known before general solutions can be written in their proper forms, e.g. equation (5.1). For different forms of the roots, the expressions for the complementary solutions and their Fourier expansions needed in the analysis are different. It should be pointed out that the Fourier expansions for the complementary solutions corresponding to only three different forms of roots were presented as given in equations (5.8), (5.10), (D.4), (D.5), (D.11) and (D.12). These are the only cases which occurred in the numerical examples considered. When other cases occur, proper Fourier expansions must be reformulated.

The method presented in this analysis is very flexible, although the numerical results are presented for only fixed-fixed supported shells. The method is applicable to cylindrical shells with any other boundary conditions, and the general analysis presented in this study is usable for shells with an arbitrary number of orthotropic layers. Of course, when the number of layers increases, the complexity and difficulty in the numerical calculations will also increase.

Several numerical examples made according to the present analysis have been presented for different values of $\ell/R$ and $h/R$ and compared to the results obtained by using the Kirchhoff hypothesis and neglecting or including the longitudinal inertia for a single-layered orthotropic cylindrical shell which is made of either barite or topaz. The differences between the frequencies obtained according to the present analysis and the analysis based on Kirchhoff assumptions with longitudinal inertia neglected are small for short shells. When the length of a shell becomes large, the differences in frequencies computed with and without longitudinal inertia and the shear deformation are significant. But, the differences between the frequencies obtained according to the present analysis and the analysis based upon the assumption of nondeformable normals only are significant for short shells, even the thickness of the orthotropic shell is relatively thin. However, for the long shell, the effect of shear deformation is negligible.

# APPENDIX A

## STRESS-STRAIN RELATIONSHIPS

Based on the assumption of transverse inextensibility, the stress-strain relationships for orthotropic materials may be written in the form [2]

$$\varepsilon_x = a_{11}\sigma_x + a_{12}\sigma_y + a_{13}\sigma_z \,, \tag{A.1}$$

$$\varepsilon_y = a_{21}\sigma_x + a_{22}\sigma_y + a_{23}\sigma_z \,,$$

$$\varepsilon_z = a_{31}\sigma_x + a_{32}\sigma_y + a_{33}\sigma_z = 0 \,,$$

$$\gamma_{yz} = a_{44}\tau_{yz} \,,$$

$$\gamma_{zx} = a_{55}\tau_{zx} \,,$$

$$\gamma_{xy} = a_{66}\tau_{xy} \,,$$

where

$$a_{11} = \frac{1}{E_1} \,, \qquad a_{12} = -\frac{\nu_{12}}{E_2} \,, \qquad a_{13} = -\frac{\nu_{13}}{E_3} \,, \tag{A.2}$$

$$a_{21} = -\frac{\nu_{21}}{E_1} \,, \quad a_{22} = \frac{1}{E_2} \,, \qquad a_{23} = -\frac{\nu_{23}}{E_3} \,,$$

$$a_{31} = -\frac{\nu_{31}}{E_1} \,, \quad a_{32} = -\frac{\nu_{32}}{E_2} \,, \qquad a_{33} = \frac{1}{E_3} \,,$$

$$a_{44} = \frac{1}{G_{23}} , \qquad a_{55} = \frac{1}{G_{13}} , \qquad a_{66} = \frac{1}{G_{12}} ,$$

and

$$E_2 \nu_{21} = E_1 \nu_{12} , \tag{A.3}$$

$$E_3 \nu_{32} = E_2 \nu_{23} ,$$

$$E_1 \nu_{13} = E_3 \nu_{31} .$$

Equations (A.3) imply that $a_{12} = a_{21}$, $a_{13} = a_{31}$ and $a_{23} = a_{32}$. Therefore, there are only nine independent elastic constants involved.

From equations (A.1), the stresses may be expressed in terms of the strains and elastic constants, i.e.

$$\sigma_x = B_{11}\varepsilon_x + B_{12}\varepsilon_y , \tag{A.4}$$

$$\sigma_y = B_{12}\varepsilon_x + B_{22}\varepsilon_y ,$$

$$\sigma_z = B_{13}\varepsilon_x + B_{23}\varepsilon_y ,$$

$$\tau_{yz} = B_{44}\gamma_{yz} ,$$

$$\tau_{zx} = B_{55}\gamma_{zx} ,$$

$$\tau_{xy} = B_{66}\gamma_{xy} ,$$

where

$$B_{11} = \frac{a^*_{22}}{\Omega} , \qquad B_{12} = -\frac{a^*_{12}}{\Omega} , \qquad B_{22} = \frac{a^*_{11}}{\Omega} , \tag{A.5}$$

$$B_{44} = \frac{1}{a_{44}} , \qquad B_{55} = \frac{1}{a_{55}} , \qquad B_{66} = \frac{1}{a_{66}} ,$$

$$B_{13} = -\frac{1}{a_{33}} (a_{13}B_{11} + a_{23}B_{12}) , \quad B_{23} = -\frac{1}{a_{33}}(a_{13}B_{12} + a_{23}B_{22}) ,$$

and where

$$a^*_{11} = a_{11} - \frac{a^2_{13}}{a_{33}} , \qquad a^*_{12} = a_{12} - \frac{a_{13}a_{23}}{a_{33}} ,$$

$$a^*_{22} = a_{22} - \frac{a^2_{23}}{a_{33}} , \qquad \Omega = a^*_{11}a^*_{22} - (a^*_{12})^2 .$$

# APPENDIX B

## GENERAL SOLUTION OF SINGLE-LAYERED SHELL

As indicated in Chapter III, the general solution for the normal displacement $w$, the longitudinal displacement $u$, and the desired shear function $\varphi$ have different forms depending on whether the roots $\lambda_k$ of the characteristic equation

$$\lambda^6 + 3s_2\lambda^4 + 3s_1\lambda^2 + s_0 = 0 \tag{B.1}$$

are real, pure imaginary, complex conjugate or multiple roots.

Letting $\xi = \lambda^2$, then the above equation reduces to

$$\xi^3 + 3s_2\xi^2 + 3s_1\xi + s_0 = 0 \; . \tag{B.2}$$

The roots of equation (B.2) may be written as [15]

$$\xi_1 = q_3 + q_4 - s_2 \; , \tag{B.3}$$

$$\xi_2 = -\frac{1}{2}(q_3 + q_4) - s_2 + \frac{i\sqrt{3}}{2}(q_3 - q_4) \; ,$$

$$\xi_3 = -\frac{1}{2}(q_3 + q_4) - s_2 - \frac{i\sqrt{3}}{2}(q_3 - q_4) \; ,$$

where

$$q_1 = s_1 - s_2^2 \tag{B.4}$$

$$q_2 = \frac{1}{2}(3s_1s_2 - s_0) - s_2^3$$

$$q_3 = [q_2 + (q_1^3 + q_2^2)^{1/2}]^{1/3} ,$$

$$q_4 = [q_2 - (q_1^3 + q_2^2)^{1/2}]^{1/3} .$$

and where $i = \sqrt{-1}$ .

In terms of the discriminant $\Delta = q_1^3 + q_2^2$, one has

(i) if $\Delta > 0$, then $\xi_1$ is real and $\xi_2$, $\xi_3$ are complex conjugate roots,

(ii) if $\Delta = 0$, then $\xi_1$, $\xi_2$, $\xi_3$ are real and $\xi_2 = \xi_3$,

(iii) if $\Delta < 0$, then all three roots are real and distinct.

The forms of solutions of W, U and $\Phi$ depend not only on the numerical values of $\Delta$, but also on the real part of $\xi_k$, i.e. whether it is positive or negative. According to the nature of the cubic equation (B.2), the following cases are discussed separately:

Case I. $\Delta > 0$

As indicated above, the characteristic equation (B.2) has one real and a pair of complex conjugate roots, which may be expressed in the forms

$$\xi_1 = b_0 , \tag{B.5}$$

$$\xi_2 = b_1 + ib_2 ,$$

$$\xi_3 = b_1 - ib_2 ,$$

where

$$b_0 = q_3 + q_4 - s_2 ,$$

$$b_1 = -\frac{1}{2}(q_3 + q_4) - s_2 ,$$

$$b_2 = \frac{\sqrt{3}}{2}(q_3 - q_4) .$$

By defining

$$b_3 = (b_1^2 + b_2^2)^{1/2}, \qquad \theta = \tan^{-1}(\frac{b_2}{b_1}) ,$$

$$\beta_o = \sqrt{|b_o|} , \qquad \alpha_1 = \sqrt{b_3} \cos\frac{\theta}{2} ,$$

$$\beta_1 = b_3 \sin\frac{\theta}{2} ,$$

the corresponding six roots of equation (B.1) can be written as

$$\lambda_{1,2} = \begin{Bmatrix} \pm \beta_o \\ \pm i\beta_o \\ 0 \end{Bmatrix} \begin{matrix} \text{if } b_0 > 0 \\ \text{if } b_0 < 0 \\ \text{if } b_0 = 0 \end{matrix}$$

$$\lambda_{3,4} = \pm(\alpha_1 + i\beta_1) \tag{B.8}$$

$$\lambda_{5,6} = \pm(\alpha_1 - i\beta_1)$$

and the corresponding solutions of U, W and $\Phi$ are

(i) if $b_0 > 0$, then

$$\overline{W} = K_1 e^{\beta_o x} + e^{\alpha_1 x}(K_2 \cos\beta_1 x + K_3 \sin\beta_1 x) + K_4 e^{-\beta_o x} \tag{B.9}$$

$$+ e^{-\alpha_1 x}(K_5 \cos\beta_1 x + K_6 \sin\beta_1 x)$$

$$U = \eta_1(K_1e^{\beta_o x} - K_4e^{-\beta_o x}) + e^{\alpha_1 x}[K_2(\eta_2 \cos \beta_1 x - \eta_3 \sin \beta_1 x)$$

$$+ K_3(\eta_3 \cos \beta_1 x + \eta_2 \sin \beta_1 x)] + e^{-\alpha_1 x}[K_5(-\eta_2 \cos \beta_1 x - \eta_3 \sin \beta_1 x)$$

$$+ K_6(\eta_3 \cos \beta_1 x - \eta_2 \sin \beta_1 x)]$$

$$\Phi = K_1\eta_4 e^{\beta_o x} + e^{\alpha_1 x}[K_2(\eta_5 \cos \beta_1 x - \eta_6 \sin \beta_1 x)$$

$$+ K_3(\eta_6 \cos \beta_1 x + \eta_5 \sin \beta_1 x)] - K_4\eta_4 e^{-\beta_o x}$$

$$+ e^{-\alpha_1 x}[K_5(-\eta_5 \cos \beta_1 x - \eta_6 \sin \beta_1 x)$$

$$+ K_6(\eta_6 \cos \beta_1 x - \eta_5 \sin \beta_1 x)]$$

(ii) if $b_0 = 0^*$, then

$$W = K_1 + e^{\alpha_1 x}(K_2 \cos \beta_1 x + K_3 \sin \beta_1 x) + K_4 x \tag{B.10}$$

$$+ e^{-\alpha_1 x}(K_5 \cos \beta_1 x + K_6 \sin \beta_1 x)$$

$$U = m_1 K_4 + e^{\alpha_1 x}[K_2(\eta_2 \cos \beta_1 x - \eta_3 \sin \beta_1 x) + K_3(\eta_3 \cos \beta_1 x + \eta_2 \sin \beta_1 x)]$$

$$+ e^{-\alpha_1 x}[K_5(-\eta_2 \cos \beta_1 x - \eta_3 \sin \beta_1 x) + K_6(\eta_3 \cos \beta_1 x - \eta_2 \sin \beta_1 x)]$$

---

*This case will rarely happen, because if one of $\xi_k$ is zero, then it implies $s_0$=0 or, equivalently, either $g_3$ or $G_2$ equals to zero. In other words, this occurs only when either $a_{55}B_{12}h^2 = 10R^2$ or $\rho\omega^2 = B_{22}\ln(\frac{2R+h}{2R-h})/Rh$. In the further discussion this case will be neglected.

$$\Phi = (m_1 f_1 + f_3)K_4 + e^{\alpha_1 x}\,[K_2(\eta_5 \cos \beta_1 x - \eta_6 \sin \beta_1 x)$$

$$+ K_3(\eta_6 \cos \beta_1 x + \eta_5 \sin \beta_1 x)] + e^{-\alpha_1 x}\,[K_5(-\eta_5 \cos \beta_1 x - \eta_6 \sin \beta_1 x)$$

$$+ K_6(\eta_6 \cos \beta_1 x - \eta_5 \sin \beta_1 x)]$$

(iii) if $b_0 < 0$, then

$$W = K_1 \cos \beta_o x + e^{\alpha_1 x}(K_2 \cos \beta_1 x + K_3 \sin \beta_1 x) \tag{B.11}$$

$$+ K_4 \sin \beta_o x + e^{-\alpha_1 x}(K_5 \cos \beta_1 x + K_6 \sin \beta_1 x)$$

$$U = K_1 \eta_1^* \sin \beta_o x + e^{\alpha_1 x}[\, K_2(\eta_2 \cos \beta_1 x - \eta_3 \sin \beta_1 x)$$

$$+ K_3(\eta_3 \cos \beta_1 x + \eta_2 \sin \beta_1 x)] - K_4 \eta_1^* \cos \beta_o x$$

$$+ e^{-\alpha_1 x}[\, K_5(-\eta_2 \cos \beta_1 x - \eta_3 \sin \beta_1 x) + K_6(\eta_3 \cos \beta_1 x - \eta_2 \sin \beta_1 x)]\,,$$

$$\Phi = K_1 \eta_4^* \sin \beta_o x + e^{\alpha_1 x}[\, K_2(\eta_5 \cos \beta_1 x - \eta_6 \sin \beta_1 x)$$

$$+ K_3(\eta_6 \cos \beta_1 x + \eta_5 \sin \beta_1 x)] - K_4 \eta_4^* \cos \beta_o x$$

$$+ e^{-\alpha_1 x}[\, K_5(-\eta_5 \cos \beta_1 x - \eta_6 \sin \beta_1 x) + K_6(\eta_6 \cos \beta_1 x - \eta_5 \sin \beta_1 x)]\,.$$

<u>Case II.</u> $\Delta = 0$

The roots of equations (B.2) and (B.5) corresponding to this case are

$$\xi_1 = b_0 , \tag{B.12}$$

$$\xi_2 = \xi_3 = b_1 ,$$

$$\lambda_{1,2} = \pm \sqrt{b_0} ,$$

and

$$\lambda_{3,4} = \lambda_{5,6} = \pm \sqrt{b_1} ,$$

$$\alpha_o = \sqrt{|b_1|} ,$$

where $\xi_k$, $k = 1,2,3$, are real. The expressions of solutions for W, U and $\Phi$ again depend on the sign of $\xi_k$. There are four cases corresponding to case II and they are

(i) if $b_0 > 0$ and $b_1 > 0$, then

$$W = Ke^{\beta_o x} + e^{\alpha_o x}(K_2 + K_3 x) + K_4 e^{-\beta_o x} + e^{-\alpha_o x}(K_5 + K_6 x), \tag{B.13}$$

$$U = \eta_1(K_1 e^{\beta_o x} - K_4 e^{-\beta_o x}) + e^{\alpha_o x}[\,\eta_7 K_2 + (\eta_8 + \eta_7 x)K_3]$$

$$+ e^{-\alpha_o x}[-\eta_7 K_5 + (\eta_8 - \eta_7 x)K_6] ,$$

$$\Phi = \eta_4(K_1 e^{\beta_o x}) + e^{\alpha_o x}[\,\eta_9 K_2 + (\eta_{10} + \eta_9 x)K_3] - K_4 \eta_4\, e^{-\beta_o x}$$

$$+ e^{-\alpha_o x}[-\eta_9 K_5 + (\eta_{10} - \eta_9 x)K_6] ,$$

(ii) if $b_0 < 0$ and $b_1 < 0$, then

$$W = K_1 \cos \beta_o x + (K_2 + K_3 x) \cos \alpha_o x + K_4 \sin \beta_o x + (K_5 + K_6 x) \sin \alpha_o x , \tag{B.14}$$

$$U = \eta_1^*(K_1 \sin \beta_o x - K_4 \cos \beta_o x) - \eta_7^* K_2 \sin \alpha_o x + K_3(\eta_8^* \cos \alpha_o x$$

$$- \eta_7^* x \sin \alpha_o x) + K_5 \eta_7^* \cos \alpha_o x + K_6(\eta_7^* x \cos \alpha_o x + \eta_8^* \sin \alpha_o x) ,$$

$$\Phi = \eta_4^*(K_1 \sin \beta_o x - K_4 \cos \beta_o x) - K_2 \eta_9^* \sin \alpha_o x + K_3(\eta_{10}^* \cos \alpha_o x$$

$$- \eta_9^* x \sin \alpha_o x) + K_5 \eta_9^* \cos \alpha_o x + K_6(\eta_{10}^* \sin \alpha_o x + \eta_9^* x \cos \alpha_o x) ,$$

(iii) if $b_0 > 0$ and $b_1 < 0$, then

$$W = K_1 e^{\beta_o x} + (K_2 + K_3 x) \cos \alpha_o x + K_4 e^{-\beta_o x} + (K_5 + K_6 x) \sin \alpha_o x , \tag{B.15}$$

$$U = \eta_1(K_1 e^{\beta_o x} - K_4 e^{-\beta_o x}) - K_2 \eta_7^* \sin \alpha_o x + K_3(\eta_8^* \cos \alpha_o x$$

$$- \eta_7^* x \sin \alpha_o x) + K_5 \eta_7^* \cos \alpha_o x + K_6(\eta_8^* \sin \alpha_o x + \eta_7^* x \cos \alpha_o x) ,$$

$$\Phi = \eta_4(K_1 e^{\beta_o x} - K_4 e^{-\beta_o x}) - K_2 \eta_9^* \sin \alpha_o x + K_3(\eta_{10}^* \cos \alpha_o x$$

$$- \eta_9^* x \sin \alpha_o x) + K_5 \eta_9^* \cos \alpha_o x + K_6(\eta_{10}^* \sin \alpha_o x + \eta_9^* x \cos \alpha_o x) .$$

(iv) if $b_0 < 0$ and $b_1 > 0$, then

$$W = K_1 \cos \beta_o x + (K_2 + K_3 x) e^{\alpha_o x} + K_4 \sin \beta_o x + (K_5 + K_6 x) e^{-\alpha_o x} , \tag{B.16}$$

$$U = \eta_1^*(K_1 \sin \beta_o x - K_4 \cos \beta_o x) + e^{\alpha_o x}[\,K_2\eta_7 + K_3(\eta_8 + \eta_7 x)]$$

$$+ e^{-\alpha_o x}[-K_5\eta_7 + K_6(\eta_8 - \eta_7 x)] \quad ,$$

$$\Phi = \eta_4^*(K_1 \sin \beta_o x - K_4 \cos \beta_o x) + e^{-\alpha_o x}[K_2\eta_9 + K_3(\eta_{10} + \eta_9 x)]$$

$$+ e^{-\alpha_o x}[-K_5\eta_9 + K_6(\eta_{10} - \eta_9 x)] \quad ,$$

<u>Case III.</u> $\Delta < 0$

The three different real roots of equation (B.2) corresponding to this case are [15]

(a) if $q_2 \geq 0$, by defining $\cos\theta = \dfrac{q_2}{\sqrt{-q_1^3}}$, then

$$\xi_1 = 2\sqrt{-q_1}\cos\frac{\theta}{3} - s_2 \,, \tag{B.17}$$

$$\xi_2 = 2\sqrt{-q_1}\cos\left(\frac{\theta}{3} + \frac{2\pi}{3}\right) - s_2$$

$$\xi_3 = 2\sqrt{-q_1}\cos\left(\frac{\theta}{3} + \frac{4\pi}{3}\right) - s_2 \,,$$

(b) if $q_2 < 0$, by defining $\cos\theta = -\dfrac{q_2}{\sqrt{-q_1^3}}$, then

$$\xi_1 = -2\sqrt{-q_1}\cos\frac{\theta}{3} - s_2 \,, \tag{B.18}$$

$$\xi_2 = -2\sqrt{-q_1}\cos\left(\frac{\theta}{3} + \frac{2\pi}{3}\right) - s_2 \,,$$

$$\xi_3 = -2\sqrt{-q_1}\cos\left(\frac{\theta}{3} + \frac{4\pi}{3}\right) - s_2 ,$$

Let

$$\gamma_k = \sqrt{|\xi_k|} \qquad \text{for } k = 1,2,3. \tag{B.19}$$

Corresponding to this case, there are eight different expressions of the solutions for W, U and $\Phi$, depending on the sign of $\xi_k$, $k = 1,2,3$.

(i) if $\xi_1 > 0$, $\xi_2 > 0$ and $\xi_3 > 0$, then

$$W = K_1 e^{\gamma_1 x} + K_2 e^{\gamma_2 x} + K_3 e^{\gamma_3 x} + K_4 e^{-\gamma_1 x} + K_5 e^{-\gamma_2 x} + K_6 e^{-\gamma_3 x}, \tag{B.20}$$

$$U = \eta_{11}(K_1 e^{\gamma_1 x} - K_4 e^{-\gamma_1 x}) + \eta_{12}(K_2 e^{\gamma_2 x} - K_5 e^{-\gamma_2 x})$$
$$+ \eta_{13}(K_3 e^{\gamma_3 x} - K_6 e^{-\gamma_3 x}) ,$$

$$\Phi = \eta_{14}(K e^{\gamma_1 x} - K_4 e^{-\gamma_1 x}) + \eta_{15}(K_2 e^{\gamma_2 x} - K_5 e^{-\gamma_2 x}) + \eta_{16}(K_3 e^{\gamma_3 x}$$
$$- K_6 e^{-\gamma_3 x}) .$$

(ii) if $\xi_1 < 0$, $\xi_2 < 0$ and $\xi_3 < 0$, then

$$W = K_1 \cos \gamma_1 x + K_2 \cos \gamma_2 x + K_3 \cos \gamma_3 x + K_4 \sin \gamma_1 x + K_5 \sin \gamma_2 x$$
$$+ K_6 \sin \gamma_3 x , \tag{B.21}$$

$$U = \eta^*_{11}(K_1 \sin \gamma_1 x - K_4 \cos \gamma_1 x) + \eta^*_{12}(K_2 \sin \gamma_2 x - K_5 \cos \gamma_2 x)$$
$$+ \eta^*_{13}(K_3 \sin \gamma_3 x - K_6 \cos \gamma_3 x) ,$$

$$\Phi = \eta_{14}^{*}(K_1 \sin \gamma_1 x - K_4 \cos \gamma_1 x) + \eta_{15}^{*}(K_2 \sin \gamma_2 x - K_5 \cos \gamma_2 x)$$

$$+ \eta_{16}^{*}(K_3 \sin \gamma_3 x - K_6 \cos \gamma_3 x) .$$

(iii) if $\xi_1 > 0$, $\xi_2 > 0$ and $\xi_3 < 0$, then

$$W = K_1 e^{\gamma_1 x} + K_2 e^{\gamma_2 x} + K_3 \cos \gamma_3 x + K_4 e^{-\gamma_1 x} + K_5 e^{-\gamma_2 x} + K_6 \sin \gamma_3 x ,$$

$$\text{(B.22)}$$

$$U = \eta_{11}(K_1 e^{\gamma_1 x} - K_4 e^{-\gamma_1 x}) + \eta_{12}(K_2 e^{\gamma_2 x} - K_5 e^{-\gamma_2 x})$$

$$+ \eta_{13}^{*}(K_3 \sin \gamma_3 x - K_6 \cos \gamma_3 x) ,$$

$$\Phi = \eta_{14}(K_1 e^{\gamma_1 x} - K_4 e^{-\gamma_1 x}) + \eta_{15}(K_2 e^{\gamma_2 x} - K_5 e^{-\gamma_2 x}) + \eta_{16}^{*}(K_3 \sin \gamma_3 x - K_6 \cos \gamma_3 x) .$$

(iv) if $\xi_1 > 0$, $\xi_2 < 0$ and $\xi_3 > 0$, then

$$W = K_1 e^{\gamma_1 x} + K_2 \cos \gamma_2 x + K_3 e^{\gamma_3 x} + K_4 e^{-\gamma_1 x} + K_5 \sin \gamma_2 x + K_6 e^{-\gamma_3 x} ,$$

$$\text{(B.23)}$$

$$U = \eta_{11}(K_1 e^{\gamma_1 x} - K_4 e^{-\gamma_1 x}) + \eta_{12}^{*}(K_2 \sin \gamma_2 x - K_5 \cos \gamma_2 x)$$

$$+ \eta_{13}(K_3 e^{\gamma_3 x} - K_6 e^{-\gamma_3 x}) ,$$

$$\Phi = \eta_{14}(K_1 e^{\gamma_1 x} - K_4 e^{-\gamma_1 x}) + \eta_{15}^{*}(K_2 \sin \gamma_2 x - K_5 \cos \gamma_2 x)$$

$$+ \eta_{16}(K_3 e^{\gamma_3 x} - K_6 e^{-\gamma_3 x}) .$$

(v) if $\xi_1 > 0$, $\xi_2 < 0$ and $\xi_3 < 0$, then

$$W = K_1 e^{\gamma_1 x} + K_2 \cos \gamma_2 x + K_3 \cos \gamma_3 x + K_4 e^{-\gamma_1 x} + K_5 \sin \gamma_2 x + K_6 \sin \gamma_3 x \ , \tag{B.24}$$

$$U = \eta_{11}(K_1 e^{\gamma_1 x} - K_4 e^{-\gamma_1 x}) + \eta_{12}^*(K_2 \sin \gamma_2 x - K_5 \cos \gamma_2 x) + \eta_{13}^*(K_3 \sin \gamma_3 x - K_6 \cos \gamma_3 x) \ ,$$

$$\Phi = \eta_{14}(K_1 e^{\gamma_1 x} - K_4 e^{-\gamma_1 x}) + \eta_{15}^*(K_2 \sin \gamma_2 x - K_5 \cos \gamma_2 x) + \eta_{16}^*(K_3 \sin \gamma_3 x - K_6 \cos \gamma_3 x) \ .$$

(vi) if $\xi_1 < 0$, $\xi_2 > 0$ and $\xi_3 > 0$, then

$$W = K_1 \cos \gamma_1 x + K_2 e^{\gamma_2 x} + K_3 e^{\gamma_3 x} + K_4 \sin \gamma_1 x + K_5 e^{-\gamma_2 x} + K_6 e^{-\gamma_3 x} \ ,$$

$$U = \eta_{11}^*(K_1 \sin \gamma_1 x - K_4 \cos \gamma_1 x) + \eta_{12}(K_2 e^{\gamma_2 x} - K_5 e^{-\gamma_2 x}) + \eta_{13}(K_3 e^{\gamma_3 x} - K_6 e^{-\gamma_3 x}) \ , \tag{B.25}$$

$$\Phi = \eta_{14}^*(K_1 \sin \gamma_1 x - K_4 \cos \gamma_1 x) + \eta_{15}(K_2 e^{\gamma_2 x} - K_5 e^{-\gamma_2 x}) + \eta_{16}(K_3 e^{\gamma_3 x} - K_6 e^{-\gamma_3 x}) \ .$$

(vii) if $\xi_1 < 0$, $\xi_2 > 0$ and $\xi_3 < 0$, then

$$W = K_1 \cos\gamma_1 x + K_2 e^{\gamma_2 x} + K_3 \cos\gamma_3 x + K_4 \sin\gamma_1 x + K_5 e^{-\gamma_2 x} \qquad (B.26)$$

$$+ K_6 \sin\gamma_3 x \ ,$$

$$U = \eta_{11}^*(K_1 \sin\gamma_1 x - K_4 \cos\gamma_1 x) + \eta_{12}(K_2 e^{\gamma_2 x} - K_5 e^{-\gamma_2 x})$$

$$+ \eta_{13}^*(K_3 \sin\gamma_3 x - K_6 \cos\gamma_3 x) \ ,$$

$$\Phi = \eta_{14}^*(K_1 \sin\gamma_1 x - K_4 \cos\gamma_1 x) + \eta_{15}(K_2 e^{\gamma_2 x} - K_5 e^{-\gamma_2 x})$$

$$+ \eta_{16}^*(K_3 \sin\gamma_3 x - K_6 \cos\gamma_3 x) \ .$$

(viii) if $\xi_1 < 0$, $\xi_2 < 0$ and $\xi_3 > 0$, then

$$W = K_1 \cos\gamma_1 x + K_2 \cos\gamma_2 x + K_3 e^{\gamma_3 x} + K_4 \sin\gamma_1 x + K_5 \sin\gamma_2 x + K_6 e^{-\gamma_3 x} ,$$

(B.27)

$$U = \eta_{11}^*(K_1 \sin\gamma_1 x - K_4 \cos\gamma_1 x) + \eta_{12}^*(K_2 \sin\gamma_2 x - K_5 \cos\gamma_2 x)$$

$$+ \eta_{13}(K_3 e^{\gamma_3 x} - K_6 e^{-\gamma_3 x}) \ ,$$

$$\Phi = \eta_{14}^*(K_1 \sin\gamma_1 x - K_4 \cos\gamma_1 x) + \eta_{15}^*(K_2 \sin\gamma_2 x - K_5 \cos\gamma_2 x)$$

$$+ \eta_{16}(K_3 e^{\gamma_3 x} - K_6 e^{-\gamma_3 x}) \ ,$$

where the constants $\eta_i$ and $\eta_i^*$ are

$$\eta_1 = m_1\beta_o + m_2\beta_o^3 + m_3\beta_o^5 , \tag{B.28}$$

$$\eta_2 = m_1\alpha_1 + m_2(\alpha_1^3 - 3\alpha_1\beta_1^2) + m_3(\alpha_1^5 - 10\alpha_1^3\beta_1^2 + 5\alpha_1\beta_1^4) ,$$

$$\eta_3 = m_1\beta_1 - m_2(\beta_1^3 - 3\alpha_1^2\beta_1) + m_3(\beta_1^5 - 10\alpha_1^2\beta_1^3 + 5\alpha_1^4\beta_1) ,$$

$$\eta_4 = f_1\eta_1 + f_2\eta_1\beta_o^2 + f_3\beta_o ,$$

$$\eta_5 = [f_1 + f_2(\alpha_1^2 - \beta_1^2)]\eta_2 - 2\alpha_1\beta_1 f_2\eta_3 + f_3\alpha_1 ,$$

$$\eta_6 = [f_1 + f_2(\alpha_1^2 - \beta_1^2)]\eta_3 + 2\alpha_1\beta_1 f_2\eta_2 + f_3\beta_1 ,$$

$$\eta_7 = m_1\alpha_o + m_2\alpha_o^3 + m_3\alpha_o^5 ,$$

$$\eta_8 = m_1 + 3m_2\alpha_o^2 + 5m_3\alpha_o^4 ,$$

$$\eta_9 = f_1\eta_7 + f_2\alpha_o^2\eta_7 + f_3\alpha_o ,$$

$$\eta_{10} = f_1\eta_8 + f_2(2\alpha_o\eta_7 + \alpha_o^2\eta_8) + f_3 ,$$

$$\eta_{11} = m_1\gamma_1 + m_2\gamma_1^3 + m_3\gamma_1^5 ,$$

$$\eta_{12} = m_1\gamma_2 + m_2\gamma_2^3 + m_3\gamma_2^5 ,$$

$$\eta_{13} = m_1\gamma_3 + m_2\gamma_3^3 + m_3\gamma_3^5 ,$$

$$\eta_{14} = (f_1 + f_2\gamma_1^2)\eta_{11} + f_3\gamma_1 \quad ,$$

$$\eta_{15} = (f_1 + f_2\gamma_2^2)\eta_{12} + f_3\gamma_2 \quad ,$$

$$\eta_{16} = (f_1 + f_2\gamma_3^2)\eta_{13} + f_3\gamma_3 \quad ,$$

$$\eta_1^* = -m_1\beta_o + m_2\beta_o^3 - m_3\beta_o^5 \quad ,$$

$$\eta_4^* = f_1\eta_1^* - f_2\beta_o^2\eta_1^* - f_3\beta_o \quad ,$$

$$\eta_7^* = m_1\alpha_o - m_2\alpha_o^3 + m_3\alpha_o^5 \quad ,$$

$$\eta_8^* = m_1 - 3m_2\alpha_o^2 + 5m_3\alpha_o^4 \quad ,$$

$$\eta_9^* = (f_1 - f_2\alpha_o^2)\eta_7^* - f_3\alpha_o \quad ,$$

$$\eta_{10}^* = f_1\eta_8^* - f_2(2\alpha_o\eta_7^* + \alpha_o^2\eta_8^*) + f_3 \quad ,$$

$$\eta_{11}^* = -m_1\gamma_1 + m_2\gamma_1^3 - m_3\gamma_1^5$$

$$\eta_{12}^* = -m_1\gamma_2 + m_2\gamma_2^3 - m_3\gamma_2^5 \quad ,$$

$$\eta_{13}^* = -m_1\gamma_3 + m_2\gamma_3^3 - m_3\gamma_3^5 \quad ,$$

$$\eta_{14}^* = (f_1 - f_2\gamma_1^2)\eta_{11}^* - f_3\gamma_1 \quad ,$$

$$\eta_{15}^* = (f_1 - f_2\gamma_2^2)\eta_{12}^* - f_3\gamma_2 \quad ,$$

$$\eta_{16}^* = (f_1 - f_2\gamma_3^2)\eta_{13}^* - f_3\gamma_3 \ .$$

If $C_4^j$ and $E_4^j$ are neglected as discussed in Chapter III in comparison with other coefficients, some of these constants have new expressions and they are

$$\eta_4 = f_1\eta_1 + f_2\eta_1\beta_o^2 + f_3\beta_o^3 \ , \tag{B.29}$$

$$\eta_5 = [f_1 + f_2(\alpha_1^2 - \beta_1^2)]\eta_2 - 2\alpha_1\beta_1 f_2\eta_3 + f_3(\alpha_1^3 - 3\alpha_1\beta_1^2) \ ,$$

$$\eta_6 = [f_1 + f_2(\alpha_1^2 - \beta_1^2)]\eta_3 + 2\alpha_1\beta_1 f_2\eta_2 + f_3(3\alpha_1^2\beta_1 - \beta_1^3) \ ,$$

$$\eta_9 = (f_1 + f_2\alpha_o^2)\eta_7 + f_3\alpha_o^3 \ ,$$

$$\eta_{10} = f_1\eta_8 + f_2(2\alpha_o\eta_7 + \alpha_o^2\eta_8) + 3f_3\alpha_o^2 \ ,$$

$$\eta_{14} = (f_1 + f_2\gamma_1^2)\eta_{11} + f_3\gamma_1^3 \ ,$$

$$\eta_{15} = (f_1 + f_2\gamma_2^2)\eta_{12} + f_3\gamma_2^3 \ ,$$

$$\eta_{16} = (f_1 + f_2\gamma_3^2)\eta_{13} + f_3\gamma_3^3 \ ,$$

$$\eta_4^* = (f_1 - f_2\beta_o^2)\eta_1^* + f_3\beta_o^3 \ ,$$

$$\eta_9^* = (f_1 - f_2\alpha_o^2)\eta_7^* + f_3\alpha_o^3 \ ,$$

$$\eta_{10}^* = f_1\eta_8^* - f_2(2\alpha_o\eta_7^* + \alpha_o^2\eta_8^*) - 3f_3\alpha_o^2 \quad ,$$

$$\eta_{14}^* = (f_1 - f_2\gamma_1^2)\eta_{11}^* + f_3\gamma_1^3 \quad ,$$

$$\eta_{15}^* = (f_1 - f_2\gamma_2^2)\eta_{12}^* + f_3\gamma_2^3 \quad ,$$

$$\eta_{16}^* = (f_1 - f_2\gamma_3^2)\eta_{13}^* + f_3\gamma_3^3 \quad .$$

# APPENDIX C

## FREQUENCY EQUATION OF FIXED-FIXED SINGLE-LAYERED SHELLS

As indicated in Chapter III, for any homogeneous boundary conditions, the frequency equation of an orthotropic cylindrical shell has different expressions depending on the form of the solutions for W, U and $\Phi$ which have been given in Appendix B. Since the geometry and the elastic properties of the whole shell system are symmetric about the middle section, for a finite length shell, if the supporting edges are also symmetric about the middle section, then it is convenient to originate the longitudinal coordinate at the middle section and to write the solutions of W, U and $\Phi$ in two parts. One part contains only even functions and the other, only odd functions. For a fixed-fixed supported orthotropic cylindrical shell, the solutions of W, U and $\Phi$ corresponding to Case I and Case III which occur most often are discussed in Appendix B and may be rewritten as follows:

Case I. $\Delta > 0$

(i) $b_0 > 0$, then

$$W = K_1 \cosh \beta_o x + K_2 \cosh \alpha_1 x \cos \beta_1 x + K_3 \sinh \alpha_1 x \sin \beta_1 x \quad (C.1)$$

$$+ K_4 \sinh \beta_o x + K_5 \sinh \alpha_1 x \cos \beta_1 x + K_6 \cosh \alpha_1 x \sin \beta_1 x,$$

$$U = \eta_1(K_1 \sinh \beta_o x + K_4 \cosh \beta_o x) + K_2(\eta_2 \sinh \alpha_1 x \cos \beta_1 x$$

$$- \eta_3 \cosh \alpha_1 x \sin \beta_1 x) + K_3(\eta_2 \cosh \alpha_1 x \sin \beta_1 x$$

$$+ \eta_3 \sinh \alpha_1 x \cos \beta_1 x) + K_5(\eta_2 \cosh \alpha_1 x \cos \beta_1 x$$

$$- \eta_3 \sinh \alpha_1 x \sin \beta_1 x) + K_6(\eta_3 \cosh \alpha_1 x \cos \beta_1 x$$

$$+ \eta_2 \sinh \alpha_1 x \sin \beta_1 x) ,$$

and

$$\Phi = K_1\eta_4 \sinh \beta_o x + K_2(\eta_5 \sinh \alpha_1 x \cos \beta_1 x - \eta_6 \cosh \alpha_1 x \sin\beta_1 x)$$

$$+ K_3(\eta_5 \cosh \alpha_1 x \sin \beta_1 x + \eta_6 \sinh \alpha_1 x \cos \beta_1 x)$$

$$+ K_4\eta_4 \cosh \beta_o x + K_5(\eta_5 \cosh \alpha_1 x \cos \beta_1 x - \eta_6 \sinh \alpha_1 x \sin\beta_1 x)$$

$$+ K_6(\eta_6 \cosh \alpha_1 x \cos \beta_1 x + \eta_5 \sinh \alpha_1 x \sin \beta_1 x) .$$

(ii) $b_0 < 0$, then

$$W = K_1 \cos \beta_o x + K_2 \cosh \alpha_1 x \cos \beta_1 x + K_3 \sinh \alpha_1 x \sin \beta_1 x \quad (C.2)$$

$$+ K_4 \sin \beta_o x + K_5 \sinh \alpha_1 x \cos \beta_1 x + K_6 \cosh \alpha_1 x \sin \beta_1 x ,$$

$$U = K_1\eta_1^* \sin \beta_o x + K_2(\eta_2 \sinh \alpha_1 x \cos \beta_1 x - \eta_3 \cosh\alpha_1 x \sin\beta_1 x)$$

$$+ K_3(\eta_2 \cosh \alpha_1 x \sin \beta_1 x + \eta_3 \sinh \alpha_1 x \cos \beta_1 x) - K_4\eta_1^* \cos\beta_o x$$

$$+ K_5(\eta_2 \cosh \alpha_1 x \cos \beta_1 x - \eta_3 \sinh \alpha_1 x \sin \beta_1 x)$$

$$+ K_6(\eta_3 \cosh \alpha_1 x \cos \beta_1 x + \eta_2 \sinh \alpha_1 x \sin \beta_1 x) ,$$

and

$$\Phi = K_1\eta_4^* \sin \beta_o x + K_2(\eta_5 \sinh \alpha_1 x \cos \beta_1 x - \eta_6 \cosh \alpha_1 x \sin \beta_1 x)$$
$$+ K_3(\eta_5 \cosh \alpha_1 x \sin \beta_1 x + \eta_6 \sinh \alpha_1 x \cos \beta_1 x) - K_4\eta_4^* \cos\beta_o x$$
$$+ K_5(\eta_5 \cosh \alpha_1 x \cos \beta_1 x - \eta_6 \sinh \alpha_1 x \sin \beta_1 x)$$
$$+ K_6(\eta_6 \cosh \alpha_1 x \cos \beta_1 x + \eta_5 \sinh \alpha_1 x \sin \beta_1 x) ,$$

<u>Case II</u>. $\Delta = 0$

(i) for $b_0 > 0$ and $b_1 > 0$, then

$$W = K_1 \cosh \beta_o x + K_2 \cosh \alpha_o x + K_3 x \sinh \alpha_o x \qquad \text{(C.3)}$$
$$+ K_4 \sinh \beta_o x + K_5 \sinh \alpha_o x + K_6 x \cosh \alpha_o x ,$$
$$U = K_1\eta_1 \sinh \beta_o x + K_2\eta_7 \sinh \alpha_o x + K_3(\eta_8 \sinh \alpha_o x$$
$$+ \eta_7 x \cosh \alpha_o x) + K_4\eta_1 \cosh \beta_o x + K_5\eta_7 \cosh \alpha_o x$$
$$+ K_6(\eta_8 \cosh \alpha_o x + \eta_7 x \sinh \alpha_o x) ,$$

and

$$\Phi = K_1\eta_4 \sinh \beta_o x + K_2\eta_9 \sinh \alpha_o x + K_3(\eta_{10} \sinh \alpha_o x$$
$$+ \eta_9 x \cosh \alpha_o x) + K_4\eta_4 \cosh \beta_o x + K_5\eta_9 \cosh \alpha_o x$$
$$+ K_6(\eta_{10} \cosh \alpha_o x + \eta_9 x \sinh \alpha_o x) .$$

The remaining forms of the solutions of Case II are essentially a combination of equations (C.3) and (B.14) depending on the signs of $b_0$ and $b_1$.

Case III. $\Delta < 0$

(i) for $\xi_1 > 0$, $\xi_2 > 0$ and $\xi_3 > 0$, then

$$W = K_1 \cosh \gamma_1 x + K_2 \cosh \gamma_2 x + K_3 \cosh \gamma_3 x + K_4 \sinh \gamma_1 x + K_5 \sinh \gamma_2 x + K_6 \sinh \gamma_3 x \tag{C.4}$$

$$U = K_1 \eta_{11} \sinh \gamma_1 x + K_2 \eta_{12} \sinh \gamma_2 x + K_3 \eta_{13} \sinh \gamma_3 x + K_4 \eta_{11} \cosh \gamma_1 x + K_5 \eta_{12} \cosh \gamma_2 x + K_6 \eta_{13} \cosh \gamma_3 x$$

and

$$\Phi = K_1 \eta_{14} \sinh \gamma_1 x + K_2 \eta_{15} \sinh \gamma_2 x + K_3 \eta_{16} \sinh \gamma_3 x + K_4 \eta_{14} \cosh \gamma_1 x + K_5 \eta_{15} \cosh \gamma_2 x + K_6 \eta_{16} \cosh \gamma_3 x$$

The remaining forms of the solutions are essentially a combination of equations (C.4) and (B.21) depending on the signs of $\xi_1$, $\xi_2$ and $\xi_3$.

The fixed boundary conditions corresponding to this new set of coordinates are

$$W\Big|_{x = \pm \frac{\ell}{2}} = 0 \tag{C.5}$$

$$U\Big|_{x = \pm \frac{\ell}{2}} = 0$$

$$\frac{dW}{dx}\Big|_{x = \pm \frac{\ell}{2}} = 0$$

## Frequency Equation

The frequency equation of fixed-fixed-supported single-layered orthotropic cylindrical shells can be divided into two classes, symmetric and antisymmetric about the mid-section.

Symmetric Case. The expression of the frequency equation for symmetric vibration still depends on the form of the solutions for displacement coordinate functions and the shear function. The frequency equations corresponding to each case are listed below.

Case I. $\Delta > 0$.

(i) For $b_0 > 0$, then the frequency equation is

$$|\bar{D}_1| = \begin{vmatrix} \cosh\frac{\beta_o \ell}{2} & c_1^* & c_2^* \\ \beta_o \sinh\frac{\beta_o \ell}{2} & \alpha_1 c_4^* - \beta_1 c_3^* & \alpha_1 c_3^* + \beta_1 c_4^* \\ \eta_1 \sinh\frac{\beta_o \ell}{2} & \eta_2 c_4^* - \eta_3 c_3^* & \eta_3 c_4^* + \eta_2 c_3^* \end{vmatrix} = 0 \qquad (C.6)$$

(ii) For $b_0 < 0$, then the frequency equation is

$$|\bar{D}_1| = \begin{vmatrix} \cos\frac{\beta_o \ell}{2} & c_1^* & c_2^* \\ -\beta_o \sin\frac{\beta_o \ell}{2} & \alpha_1 c_4^* - \beta_1 c_3^* & \alpha_1 c_3^* + \beta_1 c_4^* \\ \eta_1^* \sin\frac{\beta_o \ell}{2} & \eta_2 c_4^* - \eta_3 c_3^* & \eta_3 c_4^* + \eta_2 c_3^* \end{vmatrix} = 0 \qquad (C.7)$$

Case II. $\Delta = 0$.

(i) For $b_0 > 0$ and $b_1 > 0$, then

$$|\bar{D}_1| = \begin{vmatrix} \cosh \frac{\beta_o \ell}{2} & \cosh \frac{\alpha_o \ell}{2} & \frac{\ell}{2} \sinh \frac{\alpha_o \ell}{2} \\ \beta_o \sinh \frac{\beta_o \ell}{2} & \alpha_o \sinh \frac{\alpha_o \ell}{2} & c_5^* \\ \eta_1 \sinh \frac{\beta_o \ell}{2} & \eta_7 \sinh \frac{\alpha_o \ell}{2} & c_6^* \end{vmatrix} = 0 \; . \tag{C.8}$$

(ii) For $b_0 < 0$ and $b_1 < 0$, then

$$|\bar{D}_1| = \begin{vmatrix} \cos \frac{\beta_o \ell}{2} & \cos \frac{\alpha_o \ell}{2} & \frac{\ell}{2} \sin \frac{\alpha_o \ell}{2} \\ -\beta_o \sin \frac{\beta_o \ell}{2} & -\alpha_o \sin \frac{\alpha_o \ell}{2} & c_7^* \\ \eta_1^* \sin \frac{\beta_o \ell}{2} & -\eta_7^* \sin \frac{\alpha_o \ell}{2} & c_8^* \end{vmatrix} = 0 \; . \tag{C.9}$$

(iii) For $b_0 > 0$ and $b_1 < 0$, then

$$|\bar{D}_1| = \begin{vmatrix} \cosh \frac{\beta_o \ell}{2} & \cos \frac{\alpha_o \ell}{2} & \frac{\ell}{2} \sin \frac{\alpha_o \ell}{2} \\ \beta_o \sinh \frac{\beta_o \ell}{2} & -\alpha_o \sin \frac{\alpha_o \ell}{2} & c_7^* \\ \eta_1 \sinh \frac{\beta_o \ell}{2} & -\eta_7^* \sin \frac{\alpha_o \ell}{2} & c_8^* \end{vmatrix} = 0 \; . \tag{C.10}$$

(iv) For $b_0 < 0$ and $b_1 > 0$, then

$$|\bar{D}_1| = \begin{vmatrix} \cos\frac{\beta_o \ell}{2} & \cosh\frac{\alpha_o \ell}{2} & \frac{\ell}{2}\sinh\frac{\alpha_o \ell}{2} \\ -\beta_o \sin\frac{\beta_o \ell}{2} & \alpha_o \sinh\frac{\alpha_o \ell}{2} & c_5^* \\ \eta_1^* \sin\frac{\beta_o \ell}{2} & \eta_7 \sinh\frac{\alpha_o \ell}{2} & c_6^* \end{vmatrix} = 0. \qquad (C.11)$$

Case III. $\Delta < 0$

(i) For $\xi_1 > 0$, $\xi_2 > 0$ and $\xi_3 > 0$, then

$$|\bar{D}_1| = \begin{vmatrix} \cosh\frac{\gamma_1 \ell}{2} & \cosh\frac{\gamma_2 \ell}{2} & \cosh\frac{\gamma_3 \ell}{2} \\ \gamma_1 \sinh\frac{\gamma_1 \ell}{2} & \gamma_2 \sinh\frac{\gamma_2 \ell}{2} & \gamma_3 \sinh\frac{\gamma_3 \ell}{2} \\ \eta_{11} \sinh\frac{\gamma_1 \ell}{2} & \eta_{12} \sinh\frac{\gamma_2 \ell}{2} & \eta_{13} \sinh\frac{\gamma_3 \ell}{2} \end{vmatrix} = 0. \qquad (C.12)$$

(ii) For $\xi_1 < 0$, $\xi_2 < 0$ and $\xi_3 < 0$, then

$$|\bar{D}_1| = \begin{vmatrix} \cos\frac{\gamma_1 \ell}{2} & \cos\frac{\gamma_2 \ell}{2} & \cos\frac{\gamma_3 \ell}{2} \\ -\gamma_1 \sin\frac{\gamma_1 \ell}{2} & -\gamma_2 \sin\frac{\gamma_2 \ell}{2} & -\gamma_3 \sin\frac{\gamma_3 \ell}{2} \\ \eta_{11}^* \sin\frac{\gamma_1 \ell}{2} & \eta_{12}^* \sin\frac{\gamma_2 \ell}{2} & \eta_{13}^* \sin\frac{\gamma_3 \ell}{2} \end{vmatrix} = 0. \qquad (C.13)$$

(iii) For $\xi_1 > 0$, $\xi_2 > 0$ and $\xi_3 < 0$, then

$$|\bar{D}_1| = \begin{vmatrix} \cosh\frac{\gamma_1\ell}{2} & \cosh\frac{\gamma_2\ell}{2} & \cos\frac{\gamma_3\ell}{2} \\ \gamma_1\sinh\frac{\gamma_1\ell}{2} & \gamma_2\sinh\frac{\gamma_2\ell}{2} & -\gamma_3\sin\frac{\gamma_3\ell}{2} \\ \eta_{11}\sinh\frac{\gamma_1\ell}{2} & \eta_{12}\sinh\frac{\gamma_2\ell}{2} & \eta_{13}^*\sin\frac{\gamma_3\ell}{2} \end{vmatrix} = 0. \quad (C.14)$$

(iv) For $\xi_1 > 0$, $\xi_2 < 0$ and $\xi_3 > 0$, then

$$|\bar{D}_1| = \begin{vmatrix} \cosh\frac{\gamma_1\ell}{2} & \cos\frac{\gamma_2\ell}{2} & \cosh\frac{\gamma_3\ell}{2} \\ \gamma_1\sinh\frac{\gamma_1\ell}{2} & -\gamma_2\sin\frac{\gamma_2\ell}{2} & \gamma_3\sinh\frac{\gamma_3\ell}{2} \\ \eta_{11}\sinh\frac{\gamma_1\ell}{2} & \eta_{12}^*\sin\frac{\gamma_2\ell}{2} & \eta_{13}\sinh\frac{\gamma_3\ell}{2} \end{vmatrix} = 0. \quad (C.15)$$

(v) For $\xi_1 > 0$, $\xi_2 < 0$ and $\xi_3 < 0$, then

$$|\bar{D}_1| = \begin{vmatrix} \cosh\frac{\gamma_1\ell}{2} & \cos\frac{\gamma_2\ell}{2} & \cos\frac{\gamma_3\ell}{2} \\ \gamma_1\sinh\frac{\gamma_1\ell}{2} & -\gamma_2\sin\frac{\gamma_2\ell}{2} & -\gamma_3\sin\frac{\gamma_3\ell}{2} \\ \eta_{11}\sinh\frac{\gamma_1\ell}{2} & \eta_{12}^*\sin\frac{\gamma_2\ell}{2} & \eta_{13}^*\sin\frac{\gamma_3\ell}{2} \end{vmatrix} = 0. \quad (C.16)$$

(vi) For $\xi_1 < 0$, $\xi_2 > 0$ and $\xi_3 > 0$, then

$$|\bar{D}_1| = \begin{vmatrix} \cos\frac{\gamma_1\ell}{2} & \cosh\frac{\gamma_2\ell}{2} & \cosh\frac{\gamma_3\ell}{2} \\ -\gamma_1\sin\frac{\gamma_1\ell}{2} & \gamma_2\sinh\frac{\gamma_2\ell}{2} & \gamma_3\sinh\frac{\gamma_3\ell}{2} \\ \eta_{11}^{*}\sin\frac{\gamma_1\ell}{2} & \eta_{12}\sinh\frac{\gamma_2\ell}{2} & \eta_{13}\sinh\frac{\gamma_3\ell}{2} \end{vmatrix} = 0. \quad (C.17)$$

(vii) For $\xi_1 < 0$, $\xi_2 > 0$ and $\xi_3 < 0$, then

$$|\bar{D}_1| = \begin{vmatrix} \cos\frac{\gamma_1\ell}{2} & \cosh\frac{\gamma_2\ell}{2} & \cos\frac{\gamma_3\ell}{2} \\ -\gamma_1\sin\frac{\gamma_1\ell}{2} & \gamma_2\sinh\frac{\gamma_2\ell}{2} & -\gamma_3\sin\frac{\gamma_3\ell}{2} \\ \eta_{11}^{*}\sin\frac{\gamma_1\ell}{2} & \eta_{12}\sinh\frac{\gamma_2\ell}{2} & \eta_{13}^{*}\sin\frac{\gamma_3\ell}{2} \end{vmatrix} = 0. \quad (C.18)$$

(viii) For $\xi_1 < 0$, $\xi_2 < 0$ and $\xi_3 > 0$, then

$$|\bar{D}_1| = \begin{vmatrix} \cos\frac{\gamma_1\ell}{2} & \cos\frac{\gamma_2\ell}{2} & \cosh\frac{\gamma_3\ell}{2} \\ -\gamma_1\sin\frac{\gamma_1\ell}{2} & -\gamma_2\sin\frac{\gamma_2\ell}{2} & \gamma_3\sinh\frac{\gamma_3\ell}{2} \\ \eta_{11}^{*}\sin\frac{\gamma_1\ell}{2} & \eta_{12}^{*}\sin\frac{\gamma_2\ell}{2} & \eta_{13}\sinh\frac{\gamma_3\ell}{2} \end{vmatrix} = 0. \quad (C.19)$$

where

$$c_1^* = \cosh\frac{\alpha_1\ell}{2} \cos\frac{\beta_1\ell}{2} , \tag{C.20}$$

$$c_2^* = \sinh\frac{\alpha_1\ell}{2} \sin\frac{\beta_1\ell}{2} ,$$

$$c_3^* = \cosh\frac{\alpha_1\ell}{2} \sin\frac{\beta_1\ell}{2} ,$$

$$c_4^* = \sinh\frac{\alpha_1\ell}{2} \cos\frac{\beta_1\ell}{2} ,$$

$$c_5^* = \frac{\alpha_o\ell}{2} \cosh\frac{\alpha_o\ell}{2} + \sinh\frac{\alpha_o\ell}{2} ,$$

$$c_6^* = \eta_8 \sinh\frac{\alpha_o\ell}{2} + \eta_7 \frac{\ell}{2} \cosh\frac{\alpha_o\ell}{2} ,$$

$$c_7^* = \sin\frac{\alpha_o\ell}{2} + \frac{\alpha_o\ell}{2} \cos\frac{\alpha_o\ell}{2} ,$$

$$c_8^* = \eta_8^* \sin\frac{\alpha_o\ell}{2} + \eta_7^* \frac{\ell}{2} \cos\frac{\alpha_o\ell}{2} .$$

Antisymmetric Case. As indicated in the previous section for the symmetric case, the expression of frequency equation for antisymmetric vibration depends on the expression of the solutions for the displacement coordinate functions and the shear function. The frequency equations corresponding to each case listed in Appendix B are as follows:

Case I. $\Delta > 0$

(i) For $b_0 > 0$, then

$$|\bar{D}_2| = \begin{vmatrix} \sinh \frac{\beta_o \ell}{2} & c_4^* & c_3^* \\ \beta_o \cosh \frac{\beta_o \ell}{2} & \alpha_1 c_1^* - \beta_1 c_2^* & \alpha_1 c_2^* + \beta_1 c_1^* \\ \eta_1 \cosh \frac{\beta_o \ell}{2} & \eta_2 c_1^* - \eta_3 c_2^* & \eta_3 c_1^* + \eta_2 c_2^* \end{vmatrix} = 0 . \qquad (C.21)$$

(ii) For $b_0 < 0$, then

$$|\bar{D}_2| = \begin{vmatrix} \sin \frac{\beta_o \ell}{2} & c_4^* & c_3^* \\ \beta_o \cos \frac{\beta_o \ell}{2} & \alpha_1 c_1^* - \beta_1 c_2^* & \alpha_1 c_2^* + \beta_1 c_1^* \\ -\eta_1^* \cos \frac{\beta_o \ell}{2} & \eta_2 c_1^* - \eta_3 c_2^* & \eta_3 c_1^* + \eta_2 c_2^* \end{vmatrix} = 0 . \qquad (C.22)$$

Case II. $\Delta = 0$

(i) For $b_0 > 0$ and $b_1 > 0$, then

$$|\bar{D}_2| = \begin{vmatrix} \sinh \frac{\beta_o \ell}{2} & \sinh \frac{\alpha_o \ell}{2} & \frac{\ell}{2} \cosh \frac{\alpha_o \ell}{2} \\ \beta_o \cosh \frac{\beta_o \ell}{2} & \alpha_o \cosh \frac{\alpha_o \ell}{2} & c_9^* \\ \eta_1 \cosh \frac{\beta_o \ell}{2} & \eta_7 \cosh \frac{\alpha_o \ell}{2} & c_{10}^* \end{vmatrix} = 0 . \qquad (C.23)$$

(ii) For $b_0 < 0$ and $b_1 < 0$, then

$$|\bar{D}_2| = \begin{vmatrix} \sin \frac{\beta_o \ell}{2} & \sin \frac{\alpha_o \ell}{2} & \frac{\ell}{2} \cos \frac{\alpha_o \ell}{2} \\ \beta_o \cos \frac{\beta_o \ell}{2} & \alpha_o \cos \frac{\alpha_o \ell}{2} & c_{11}^* \\ -\eta_1^* \cos \frac{\beta_o \ell}{2} & \eta_7^* \cos \frac{\alpha_o \ell}{2} & c_{12}^* \end{vmatrix} = 0 . \qquad \text{(C.24)}$$

(iii) For $b_0 > 0$ and $b_1 < 0$, then

$$|\bar{D}_2| = \begin{vmatrix} \sinh \frac{\beta_o \ell}{2} & \sin \frac{\alpha_o \ell}{2} & \frac{\ell}{2} \cos \frac{\alpha_o \ell}{2} \\ \beta_o \cosh \frac{\beta_o \ell}{2} & \alpha_o \cos \frac{\alpha_o \ell}{2} & c_{11}^* \\ \eta_1 \cosh \frac{\beta_o \ell}{2} & \eta_7^* \cos \frac{\alpha_o \ell}{2} & c_{12}^* \end{vmatrix} = 0 . \qquad \text{(C.25)}$$

(iv) For $b_0 < 0$ and $b_1 > 0$, then

$$|\bar{D}_2| = \begin{vmatrix} \sin \frac{\beta_o \ell}{2} & \sinh \frac{\alpha_o \ell}{2} & \frac{\ell}{2} \cosh \frac{\alpha_o \ell}{2} \\ \beta_o \cos \frac{\beta_o \ell}{2} & \alpha_o \cosh \frac{\alpha_o \ell}{2} & c_9^* \\ -\eta_1^* \cos \frac{\beta_o \ell}{2} & \eta_7 \cosh \frac{\alpha_o \ell}{2} & c_{10}^* \end{vmatrix} = 0 . \qquad \text{(C.26)}$$

Case III. $\Delta < 0$

(i) For $\xi_1 > 0$, $\xi_2 > 0$ and $\xi_3 > 0$, then

$$|\bar{D}_2| = \begin{vmatrix} \sinh \frac{\gamma_1 \ell}{2} & \sinh \frac{\gamma_2 \ell}{2} & \sinh \frac{\gamma_3 \ell}{2} \\ \gamma_1 \cosh \frac{\gamma_1 \ell}{2} & \gamma_2 \cosh \frac{\gamma_2 \ell}{2} & \gamma_3 \cosh \frac{\gamma_3 \ell}{2} \\ \eta_{11} \cosh \frac{\gamma_1 \ell}{2} & \eta_{12} \cosh \frac{\gamma_2 \ell}{2} & \eta_{13} \cosh \frac{\gamma_3 \ell}{2} \end{vmatrix} = 0. \quad \text{(C.27)}$$

(ii) For $\xi_1 < 0$, $\xi_2 < 0$ and $\xi_3 < 0$, then

$$|\bar{D}_2| = \begin{vmatrix} \sin \frac{\gamma_1 \ell}{2} & \sin \frac{\gamma_2 \ell}{2} & \sin \frac{\gamma_3 \ell}{2} \\ \gamma_1 \cos \frac{\gamma_1 \ell}{2} & \gamma_2 \cos \frac{\gamma_2 \ell}{2} & \gamma_3 \cos \frac{\gamma_3 \ell}{2} \\ -\eta_{11}^* \cos \frac{\gamma_1 \ell}{2} & -\eta_{12}^* \cos \frac{\gamma_2 \ell}{2} & -\eta_{13}^* \cos \frac{\gamma_3 \ell}{2} \end{vmatrix} = 0. \quad \text{(C.28)}$$

(iii) For $\xi_1 > 0$, $\xi_2 > 0$ and $\xi_3 < 0$, then

$$|\bar{D}_2| = \begin{vmatrix} \sinh \frac{\gamma_1 \ell}{2} & \sinh \frac{\gamma_2 \ell}{2} & \sin \frac{\gamma_3 \ell}{2} \\ \gamma_1 \cosh \frac{\gamma_1 \ell}{2} & \gamma_2 \cosh \frac{\gamma_2 \ell}{2} & \gamma_3 \cos \frac{\gamma_3 \ell}{2} \\ \eta_{11} \cosh \frac{\gamma_1 \ell}{2} & \eta_{12} \cosh \frac{\gamma_2 \ell}{2} & -\eta_{13}^* \cos \frac{\gamma_3 \ell}{2} \end{vmatrix} = 0. \quad \text{(C.29)}$$

(iv) For $\xi_1 > 0$, $\xi_2 < 0$ and $\xi_3 > 0$, then

$$|\bar{D}_2| = \begin{vmatrix} \sinh \frac{\gamma_1 \ell}{2} & \sin \frac{\gamma_2 \ell}{2} & \sinh \frac{\gamma_3 \ell}{2} \\ \gamma_1 \cosh \frac{\gamma_1 \ell}{2} & \gamma_2 \cos \frac{\gamma_2 \ell}{2} & \gamma_3 \cosh \frac{\gamma_3 \ell}{2} \\ \eta_{11} \cosh \frac{\gamma_1 \ell}{2} & -\eta_{12}^* \cos \frac{\gamma_2 \ell}{2} & \eta_{13} \cosh \frac{\gamma_3 \ell}{2} \end{vmatrix} = 0 . \quad (C.30)$$

(v) For $\xi_1 > 0$, $\xi_2 < 0$ and $\xi_3 < 0$, then

$$|\bar{D}_2| = \begin{vmatrix} \sinh \frac{\gamma_1 \ell}{2} & \sin \frac{\gamma_2 \ell}{2} & \sin \frac{\gamma_3 \ell}{2} \\ \gamma_1 \cosh \frac{\gamma_1 \ell}{2} & \gamma_2 \cos \frac{\gamma_2 \ell}{2} & \gamma_3 \cos \frac{\gamma_3 \ell}{2} \\ \eta_{11} \cosh \frac{\gamma_1 \ell}{2} & -\eta_{12}^* \cos \frac{\gamma_2 \ell}{2} & -\eta_{13}^* \cos \frac{\gamma_3 \ell}{2} \end{vmatrix} = 0 . \quad (C.31)$$

(vi) For $\xi_1 < 0$, $\xi_2 > 0$ and $\xi_3 > 0$, then

$$|\bar{D}_2| = \begin{vmatrix} \sin \frac{\gamma_1 \ell}{2} & \sinh \frac{\gamma_2 \ell}{2} & \sinh \frac{\gamma_3 \ell}{2} \\ \gamma_1 \cos \frac{\gamma_1 \ell}{2} & \gamma_2 \cosh \frac{\gamma_2 \ell}{2} & \gamma_3 \cosh \frac{\gamma_3 \ell}{2} \\ -\eta_{11}^* \cos \frac{\gamma_1 \ell}{2} & \eta_{12} \cosh \frac{\gamma_2 \ell}{2} & \eta_{13} \cosh \frac{\gamma_3 \ell}{2} \end{vmatrix} = 0 . \quad (C.32)$$

(vii) For $\xi_1 < 0$, $\xi_2 > 0$ and $\xi_3 < 0$, then

$$|\bar{D}_2| = \begin{vmatrix} \sin\frac{\gamma_1 \ell}{2} & \sinh\frac{\gamma_2 \ell}{2} & \sin\frac{\gamma_3 \ell}{2} \\ \gamma_1 \cos\frac{\gamma_1 \ell}{2} & \gamma_2 \cosh\frac{\gamma_2 \ell}{2} & \gamma_3 \cos\frac{\gamma_3 \ell}{2} \\ -\eta_{11}^{*} \cos\frac{\gamma_1 \ell}{2} & \eta_{12} \cosh\frac{\gamma_2 \ell}{2} & -\eta_{13}^{*} \cos\frac{\gamma_3 \ell}{2} \end{vmatrix} = 0 . \quad (C.33)$$

(viii) For $\xi_1 < 0$, $\xi_2 < 0$ and $\xi_3 > 0$, then

$$|\bar{D}_2| = \begin{vmatrix} \sin\frac{\gamma_1 \ell}{2} & \sin\frac{\gamma_2 \ell}{2} & \sinh\frac{\gamma_3 \ell}{2} \\ \gamma_1 \cos\frac{\gamma_1 \ell}{2} & \gamma_2 \cos\frac{\gamma_2 \ell}{2} & \gamma_3 \cosh\frac{\gamma_3 \ell}{2} \\ -\eta_{11}^{*} \cos\frac{\gamma_1 \ell}{2} & -\eta_{12}^{*} \cos\frac{\gamma_2 \ell}{2} & \eta_{13} \cosh\frac{\gamma_3 \ell}{2} \end{vmatrix} = 0 . \quad (C.34)$$

where

$$c_9^{*} = \cosh\frac{\alpha_o \ell}{2} + \frac{\alpha_o \ell}{2} \sinh\frac{\alpha_o \ell}{2} , \quad (C.35)$$

$$c_{10}^{*} = \eta_8 \cosh\frac{\alpha_o \ell}{2} + \eta_7 \frac{\ell}{2} \sinh\frac{\alpha_o \ell}{2} ,$$

$$c_{11}^{*} = \cos\frac{\alpha_o \ell}{2} - \frac{\alpha_o \ell}{2} \sin\frac{\alpha_o \ell}{2} ,$$

$$c_{12}^{*} = \eta_8^{*} \cos\frac{\alpha_o \ell}{2} - \eta_7^{*} \frac{\ell}{2} \sin\frac{\alpha_o \ell}{2} .$$

# APPENDIX D

## INFORMATION RELATED TO THE ILLUSTRATIVE EXAMPLE GIVEN IN CHAPTER V

For the shell dimensions used in the computation of the numerical examples for a single-layered cylindrical shell, which is made of either barite or topaz, i.e. $h = 0.06$ inches, $R = 6$ inches, when $\bar{\lambda}^{(1)} < 1.68675$ or $\bar{\lambda}^{(2)} < 2.05608$, the complementary solution of equation (4.1) has the form given in equation (5.1). When $1.68675 < \bar{\lambda}^{(1)} < 2.52130$ or $2.05608 < \bar{\lambda}^{(2)} < 2.34396$ then $\Delta < 0$ and $\xi_1 < 0$, $\xi_2 > 0$ and $\xi_3 < 0$. When $\bar{\lambda}^{(1)} > 2.52130$ or $\bar{\lambda}^{(2)} > 2.34396$ then $\Delta < 0$ and $\xi_1 > 0$, $\xi_2 < 0$ and $\xi_3 < 0$.

### The Case $\Delta < 0$ and $\xi_1 < 0$, $\xi_2 > 0$ and $\xi_3 < 0$

The solutions of $W^j$, $U^j$ and $\Phi^j$ corresponding to this case have the expressions

$$W^j = K_1^j \cos \gamma_1^j x + K_2^j e^{\gamma_2^j x} + K_3^j \cos \gamma_3^j x + K_4^j \sin \gamma_1^j x \qquad (D.1)$$

$$+ K_5^j e^{-\gamma_2^j x} + K_6^j \sin \gamma_3^j x + \sum_{n=0}^{\infty} T_n^j (A_{1n}^j b_n^j + A_{2n}^j b_n^{j-1}$$

$$+ A_{3n}^j d_n^j + A_{4n}^j d_n^{j-1}) \cos \frac{n\pi x}{\ell} ,$$

$$U^j = K_1^j \eta_{11}^{*j} \sin \gamma_1^j x + K_2^j \eta_{12}^j e^{\gamma_2^j x} + K_3^j \eta_{13}^{*j} \sin \gamma_3^j x \tag{D.2}$$

$$- K_4^j \eta_{11}^{*j} \cos \gamma_1^j x - K_5^j \eta_{12}^j e^{-\gamma_2^j x} - K_6^j \eta_{13}^{*} \cos \gamma_3^j x$$

$$+ \sum_{n=1}^{\infty} (\bar{a}_{1n}^j b_n^j + \bar{a}_{2n}^j b_n^{j-1} + \bar{a}_{3n}^j d_n^j + \bar{a}_{4n}^j d_n^{j-1}) \sin \frac{n\pi x}{\ell} ,$$

and

$$\Phi^j = K_1^j \eta_{14}^{*j} \sin \gamma_1^j x + K_2^j \eta_{15}^j e^{\gamma_2^j x} + K_3^j \eta_{16}^{*j} \sin \gamma_3^j x \tag{D.3}$$

$$- K_4^j \eta_{14}^{*j} \cos \gamma_1^j x - K_5^j \eta_{15}^j e^{-\gamma_2^j x} - K_6^j \eta_{16}^{*j} \cos \gamma_3^j x$$

$$+ \sum_{n=1}^{\infty} (\bar{g}_{1n}^j b_n^j + \bar{g}_{2n}^j b_n^{j-1} + \bar{g}_{3n}^j d_n^j + \bar{g}_{4n}^j d_n^{j-1}) \sin \frac{n\pi x}{\ell} .$$

The coefficients of the Fourier cosine series corresponding to this case are

$$t_{1n}^j = \frac{(-1)^n \, 2\gamma_1^j \sin \gamma_1^j \ell}{\ell[\gamma_1^{j^2} - (\frac{n\pi}{\ell})^2]} , \tag{D.4}$$

$$t_{2n}^j = \frac{2(\frac{n\pi}{\ell})[1 - (-1)^n \, e^{\gamma_2^j \ell}]}{\ell[\gamma_2^{j^2} + (\frac{n\pi}{\ell})^2]} ,$$

$$t_{3n}^j = \frac{(-1)^n \, 2\gamma_3^j \sin \gamma_3^j \ell}{\ell[\gamma_3^{j^2} - (\frac{n\pi}{\ell})^2]} ,$$

$$t^j_{4n} = \frac{2[\gamma^j_1 + (-1)^{n+1}\,\gamma^j_1 \cos\gamma^j_1 \ell]}{\ell[\gamma^{j^2}_1 - (\frac{n\pi}{\ell})^2]},$$

$$t^j_{5n} = \frac{2(\frac{n\pi}{\ell})[1 - e^{-\gamma^j_2 \ell}(-1)^n]}{\ell[\gamma^{j^2}_2 + (\frac{n\pi}{\ell})^2]},$$

$$t^j_{6n} = \frac{2\gamma^j_3[1 + (-1)^{n+1} \cos\gamma^j_3 \ell]}{\ell[\gamma^{j^2}_3 - (\frac{n\pi}{\ell})^2]},$$

for $n = 1,2,3,\ldots.$ For $t_{\mu 0}$ terms where $\mu = 1,2,3,\ldots,6$, one may substitute $n = 0$ into the right hand side of equation (D.4) and divide it by 2.

Similarly, the coefficients of the Fourier sine series corresponding to this case are

$$r^j_{1n} = -\frac{2(\frac{n\pi}{\ell})[1 - (-1)^n \cos\gamma^j_1 \ell]}{\ell[\gamma^{j^2}_1 - (\frac{n\pi}{\ell})^2]}, \qquad \text{(D.5)}$$

$$r^j_{2n} = \frac{2\gamma^j_2[(-1)^n\, e^{\gamma^j_2 \ell} - 1]}{\ell[\gamma^{j^2}_2 + (\frac{n\pi}{\ell})^2]},$$

$$r^j_{3n} = -\frac{2(\frac{n\pi}{\ell})[1 - (-1)^n \cos\gamma^j_3 \ell\ ]}{\ell[\gamma^{j^2}_3 - (\frac{n\pi}{\ell})^2]},$$

$$r^j_{4n} = \frac{2(-1)^n(\frac{n\pi}{\ell})\ \sin\gamma^j_1 \ell}{\ell[\gamma^{j^2}_1 - (\frac{n\pi}{\ell})^2]},$$

$$r_{5n}^{j} = \frac{2\gamma_2^j[1 - (-1)^n e^{-\gamma_2^j \ell}]}{\ell[\gamma_2^{j^2} + (\frac{n\pi}{\ell})^2]} ,$$

$$r_{6n}^{j} = \frac{2(-1)^n(\frac{n\pi}{\ell}) \sin \gamma_3^j \ell}{\ell[\gamma_3^{j^2} - (\frac{n\pi}{\ell})^2]} .$$

The matrix $[\bar{\xi}]^j$ of equation (5.14) has the form

$$[\bar{\xi}]^j = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & \gamma_2 & 0 & \gamma_1 & -\gamma_2 & \gamma_3 \\ 0 & \eta_{12} & 0 & -\eta_{11}^* & -\eta_{12} & -\eta_{13}^* \\ \cos \gamma_1 \ell & e^{\gamma_2 \ell} & \cos \gamma_3 \ell & \sin \gamma_1 \ell & e^{-\gamma_2 \ell} & \sin \gamma_3 \ell \\ -\gamma_1 \sin \gamma_1 \ell & \gamma_2 e^{\gamma_2 \ell} & -\gamma_3 \sin \gamma_3 \ell & \gamma_1 \cos \gamma_1 \ell & -\gamma_2 e^{-\gamma_2 \ell} & \gamma_3 \cos \gamma_3 \ell \\ \eta_{11}^* \sin \gamma_1 \ell & \eta_{12} e^{\gamma_2 \ell} & \eta_{13}^* \sin \gamma_3 \ell & -\eta_{11}^* \cos \gamma_1 \ell & -\eta_{12} e^{-\gamma_2 \ell} & -\eta_{13}^* \cos \gamma_3 \ell \end{bmatrix}^{j^{-1}}$$

(D.6)

and the coefficients $a_{n\nu}^{*j}$, and $J_{n\nu}^{*j}$ of equation (5.17) become

$$a_{n\nu}^{*j} = -T_\nu^j \Big\{ \eta_{11}^{*j} r_{4n}^j[\xi_{11}^j + (-1)^\nu \xi_{14}^j] + \eta_{12}^j r_{2n}^j[\xi_{21}^j + (-1)^\nu \xi_{24}^j] \quad \text{(D.7)}$$

$$+ \eta_{13}^{*j} r_{6n}^j[\xi_{31}^j + (-1)^\nu \xi_{34}^j] - \eta_{11}^{*j} r_{1n}^j[\xi_{41}^j + (-1)^\nu \xi_{44}^j]$$

$$- \eta_{12}^j r_{5n}^j[\xi_{51}^j + (-1)^\nu \xi_{54}^j] - \eta_{13}^{*j} r_{3n}^j[\xi_{61}^j + (-1)^\nu \xi_{64}^j] \Big\} ,$$

$$J^{*j}_{n\nu} = - T^j_\nu \Big\{ \eta^{*j}_{14}\, r^j_{4n}[\xi^j_{11} + (-1)^\nu\, \xi^j_{14}] + \eta^j_{15} r^j_{2n}[\xi^j_{21} + (-1)^\nu \xi^j_{24}]$$

$$+ \eta^{*j}_{16}\, r^j_{6n}[\xi^j_{31} + (-1)^\nu\, \xi^j_{34}] - \eta^{*j}_{14}\, r^j_{1n}[\xi^j_{41} + (-1)^\nu\, \xi^j_{44}]$$

$$- \eta^j_{15}\, r^j_{5n}[\xi^j_{51} + (-1)^\nu \xi^j_{54}] - \eta^{*j}_{16}\, r^j_{3n}[\xi^j_{61} + (-1)^\nu\, \xi^j_{64}] \Big\} .$$

<u>The Case $\Delta < 0$ and $\xi_1 > 0$, $\xi_2 < 0$ and $\xi_3 < 0$</u>

The solutions of $W^j$, $U^j$ and $\Phi^j$ corresponding to this case have the expressions

$$W^j = K^j_1 e^{\gamma^j_1 x} + K^j_2 \cos \gamma^j_2 x + K^j_3 \cos \gamma^j_3 x + K^j_4\, e^{-\gamma^j_1 x} + K^j_5 \sin \gamma^j_2 x \quad \text{(D.8)}$$

$$+ K^j_6 \sin \gamma^j_3 x + \sum_{n=0}^{\infty} T^j_n (A^j_{1n} b^j_n + A^j_{2n} b^{j-1}_n + A^j_{3n} d^j_n$$

$$+ A^j_{4n} d^{j-1}_n) \cos \frac{n\pi x}{\ell} ,$$

$$U^j = \eta^j_{11}(K^j_1\, e^{\gamma^j_1 x} - K^j_4\, e^{-\gamma^j_1 x}) + \eta^{*j}_{12}(K^j_2 \sin \gamma^j_2 x - K^j_5 \cos \gamma^j_2 x) \quad \text{(D.9)}$$

$$+ \eta^{*j}_{13}(K^j_3 \sin \gamma^j_3 x - K^j_6 \cos \gamma^j_3 x) + \sum_{n=1}^{\infty} (\bar{a}^j_{1n} b^j_n + \bar{a}^j_{2n} b^{j-1}_n$$

$$+ \bar{a}^j_{3n} d^j_n + \bar{a}^j_{4n} d^{j-1}_n) \sin \frac{n\pi x}{\ell}$$

and

$$\Phi^j = \eta^j_{14}(K^j_1\, e^{\gamma^j_1 x} - K^j_4\, e^{-\gamma^j_1 x}) + \eta^{*j}_{15}(K^j_2 \sin \gamma^j_2 x - K^j_5 \cos \gamma^j_2 x) \quad \text{(D.10)}$$

$$+ \eta_{16}^{*j}(K_3^j \sin \gamma_3^j x - K_6^j \cos \gamma_3^j x) + \sum_{n=1}^{\infty} (\bar{g}_{1n}^j b_n^j + \bar{g}_{2n}^j b_n^{j-1}$$

$$+ \bar{g}_{3n}^j d_n^j + \bar{g}_{4n}^j d_n^{j-1}) \sin \frac{n\pi x}{\ell} \quad .$$

The coefficients of the Fourier cosine series corresponding to this case are

$$t_{1n}^j = \frac{2(\frac{n\pi}{\ell})[1 - (-1)^n e^{\gamma_1^j \ell}]}{\ell[\gamma_1^{j^2} + (\frac{n\pi}{\ell})^2]}, \tag{D.11}$$

$$t_{2n}^j = \frac{2(-1)^n \gamma_2^j \sin \gamma_2^j \ell}{\ell[\gamma_2^{j^2} - (\frac{n\pi}{\ell})^2]} \quad ,$$

$$t_{4n}^j = \frac{2(\frac{n\pi}{\ell})[1 - (-1)^n e^{-\gamma_1^j \ell}]}{\ell[\gamma_1^{j^2} + (\frac{n\pi}{\ell})^2]} \quad ,$$

$$t_{5n}^j = \frac{2[\gamma_2^j + (-1)^{n+1} \gamma_2^j \cos \gamma_2^j \ell]}{\ell[\gamma_2^{j^2} - (\frac{n\pi}{\ell})^2]} \ .$$

where $t_{3n}^j$ and $t_{6n}^j$ are the same as those given in equation (D.4). The $t_{\mu 0}^j$ terms may be obtained in the same manner as those in the previous section.

The coefficients of the Fourier sine series corresponding to this case are

$$r_{1n}^{j} = \frac{2\gamma_1^j[(-1)^n e^{\gamma_1^j \ell} - 1]}{\ell[\gamma_1^{j^2} + (\frac{n\pi}{\ell})^2]}\,, \tag{D.12}$$

$$r_{2n}^{j} = -\frac{2(\frac{n\pi}{\ell})[1 - (-1)^n \cos \gamma_2^j \ell]}{\ell[\gamma_2^{j^2} - (\frac{n\pi}{\ell})^2]}\,,$$

$$r_{4n}^{j} = \frac{2\gamma_1^j[1 - (-1)^n e^{-\gamma_1^j \ell}]}{\ell[\gamma_1^{j^2} + (\frac{n\pi}{2})^2]}\,,$$

$$r_{5n}^{j} = \frac{2(-1)^n(\frac{n\pi}{\ell}) \sin \gamma_2^j \ell}{\ell[\gamma_2^{j^2} - (\frac{n\pi}{\ell})^2]}\,.$$

The matrix $[\bar{\xi}]^j$ of equation (5.14) has the form

$$[\bar{\xi}]^j = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ \gamma_1 & 0 & 0 & -\gamma_1 & \gamma_2 & \gamma_3 \\ \eta_{11} & 0 & 0 & -\eta_{11} & -\eta_{12}^* & -\eta_{13}^* \\ e^{\gamma_1 \ell} & \cos \gamma_2 \ell & \cos \gamma_3 \ell & e^{-\gamma_1 \ell} & \sin \gamma_2 \ell & \sin \gamma_3 \ell \\ \gamma_1 e^{\gamma_1 \ell} & -\gamma_2 \sin \gamma_2 \ell & -\gamma_3 \sin \gamma_3 \ell & -\gamma_1 e^{-\gamma_1 \ell} & \gamma_2 \cos \gamma_2 \ell & \gamma_3 \cos \gamma_3 \ell \\ \eta_{11} e^{\gamma_1 \ell} & \eta_{12}^* \sin \gamma_2 \ell & \eta_{13}^* \sin \gamma_3 \ell & -\eta_{11} e^{-\gamma_1 \ell} & -\eta_{12}^* \cos \gamma_2 \ell & -\eta_{13}^* \cos \gamma_3 \ell \end{bmatrix}^{j\,-1} \tag{D.13}$$

and the coefficients $a_{n\nu}^{*j}$, and $J_{n\nu}^{*j}$ of equation (5.17) become

$$\alpha_{n\nu}^{*j} = -T_\nu^j \Big\{ \eta_{11}^j r_{4n}^j [\xi_{11}^j + (-1)^\nu \xi_{14}^j] + \eta_{12}^{*j} r_{2n}^j [\xi_{21}^j + (-1)^\nu \xi_{24}^j] \tag{D.14}$$

$$+ \eta_{13}^{*j} r_{6n}^j [\xi_{31}^j + (-1)^\nu \xi_{34}^j] - \eta_{11}^j r_{1n}^j [\xi_{41}^j + (-1)^\nu \xi_{44}^j]$$

$$- \eta_{12}^{*j} r_{5n}^j [\xi_{51}^j + (-1)^\nu \xi_{54}^j] - \eta_{13}^{*j} r_{3n}^j [\xi_{61}^j + (-1)^\nu \xi_{64}^j] \Big\}$$

$$J_{n\nu}^{*j} = -T_\nu^j \Big\{ \eta_{14}^j r_{4n}^j [\xi_{11}^j + (-1)^\nu \xi_{14}^j] + \eta_{15}^{*j} r_{2n}^j [\xi_{21}^j + (-1)^\nu \xi_{24}^j]$$

$$+ \eta_{16}^{*j} r_{6n}^j [\xi_{31}^j + (-1)^\nu \xi_{34}^j] - \eta_{14}^j r_{1n}^j [\xi_{41}^j + (-1)^\nu \xi_{44}^j]$$

$$- \eta_{15}^{*j} r_{5n}^j [\xi_{51}^j + (-1)^\nu \xi_{54}^j] - \eta_{16}^{*j} r_{3n}^j [\xi_{61}^j + (-1)^\nu \xi_{64}^j] \Big\}$$

# APPENDIX E

## ROTATIONALLY SYMMETRIC VIBRATION OF A SINGLE-LAYERED ORTHOTROPIC CYLINDRICAL SHELL BASED ON CLASSICAL ASSUMPTIONS

As indicated by Ambartsumian [13], the accuracy of the hypothesis of nondeformable normals determined for isotropic shells is often quite unacceptable for anisotropic shells even in the case where the relative thickness h/R is not sufficiently large. In this analysis, the transverse shear deformation and longitudinal inertia are included. In order to determine how the results are affected by the assumption of nondeformable normals, the same problem is considered in this appendix excluding the shear deformation. Two cases are presented, the first case includes longitudinal inertia and the second case excludes longitudinal inertia.

### Classical Analysis with Longitudinal Inertia Excluded

Based on the bending theory of thin shells with the additional assumption of negligible longitudinal inertia and with $T_x = 0$, the equations of motion may be written as [31]

$$T_x = 0 \tag{E.1}$$

$$\frac{\partial N_x}{\partial x} - \frac{1}{R} T_y + q_n = \rho h \frac{\partial^2 w}{\partial t^2} \tag{E.2}$$

$$\frac{\partial M_x}{\partial x} - N_x = 0 \tag{E.3}$$

Equations (E.2) and (E.3) can be combined to obtain

$$\frac{\partial^2 M_x}{\partial x^2} - \frac{T_y}{R} + q_n = \rho h \frac{\partial^2 w}{\partial t^2} \tag{E.4}$$

For the case of rotationally symmetric vibration, when $T_x$, $T_y$ and $M_x$ are expressed in terms of displacement components, they have the form

$$T_x = B_{11} h \frac{\partial u_o}{\partial x} + \frac{h\, B_{12}}{R} w \tag{E.5}$$

$$T_y = B_{12} h \frac{\partial u_o}{\partial x} + \frac{hB_{22}}{R} w \tag{E.6}$$

$$M_x = - \frac{B_{11} h^3}{12} \frac{\partial^2 w}{\partial x^2} \tag{E.7}$$

From equations (E.1) and (E.5), one obtains

$$\frac{\partial u_o}{\partial x} = - \frac{B_{12}}{B_{11} R} w \tag{E.8}$$

The substitution of equation (E.8) into equation (E.6) results in the expression

$$T_y = \left(- \frac{B_{12}^2}{B_{11}} \frac{h}{R} + \frac{B_{22} h}{R}\right) w \tag{E.9}$$

The governing differential equation of motion then is obtained by substituting equations (E.7) and (E.9) into equation (E.4). It reads

$$\frac{d^4w}{dx^4} + C^*w = 0 \tag{E.10}$$

where

$$C^* = \frac{12}{h^2}\left[ \frac{1}{R^2}\left(\frac{B_{22}}{B_{11}} - \frac{B_{12}^2}{B_{11}^2}\right) - \frac{\rho\omega^2}{B_{11}} \right] . \tag{E.11}$$

Case I. If $C^* > 0$, then by letting $4\beta^4 = C^*$, one can write the solution of equation (E.10) as follows:

$$w = k_1 \cosh \beta x \cos \beta x + k_2 \sinh \beta x \sin \beta x + k_3 \cosh \beta x \sin \beta x + k_4 \sinh \beta x \cos \beta x . \tag{E.12}$$

Case II. If $C^* < 0$, one may define $\lambda^4 = -C^*$, with the solution

$$w = k_1 \cosh \lambda x + k_2 \cos \lambda x + K_3 \sinh \lambda x + k_4 \sin \lambda x . \tag{E.13}$$

For clamped-clamped supported edges, the boundary conditions are

$$w = w_x = 0 \qquad \text{at} \quad x = \pm \frac{\ell}{2} . \tag{E.14}$$

The frequency equations corresponding to both cases are

Case I.

$$\cosh \frac{\beta\ell}{2} \sinh \frac{\beta\ell}{2} + \sin \frac{\beta\ell}{2} \cos \frac{\beta\ell}{2} = 0 \tag{E.15}$$

for symmetrical vibration, and

$$\cosh \frac{\beta \ell}{2} \sinh \frac{\beta \ell}{2} - \sin \frac{\beta \ell}{2} \cos \frac{\beta \ell}{2} = 0 \quad \text{(E.16)}$$

for antisymmetrical vibration.

Case II.

$$\tanh \frac{\lambda \ell}{2} + \tan \frac{\lambda \ell}{2} = 0 \quad \text{(E.17)}$$

for symmetrical vibration, and

$$\tanh \frac{\lambda \ell}{2} - \tan \frac{\lambda \ell}{2} = 0 \quad \text{(E.18)}$$

for antisymmetrical vibration.

For the materials used in the numerical calculation in this analysis, when $\ell$ is less than 0.89811/R for topaz and 0.67676/R for barite respectively, equation (E.13) is the solution of differential equation (E.10). When $\ell$ is greater than those values respectively, equation (E.12) must be used instead of equation (E.13).

## Classical Analysis with Longitudinal Inertia Included

If longitudinal inertia of the shell is included in the analysis, the governing differential equations of motion in terms of displacements are [2]

$$C_{11} \frac{\partial^2 u_o}{\partial x^2} + \frac{C_{12}}{R} \frac{\partial w}{\partial x} - K_{11} \frac{\partial^3 w}{\partial x^3} - \rho h \frac{\partial^2 u_o}{\partial x^2} = 0 \quad \text{(E.19)}$$

$$\frac{C_{12}}{R} \frac{\partial u_o}{\partial x} - K_{11} \frac{\partial^3 u_o}{\partial x^3} + \frac{C_{22}}{R^2} w + \frac{2K_{12}}{R} \frac{\partial^2 w}{\partial x^2} + D_{11} \frac{\partial^4 w}{\partial x^4} + \rho h \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{(E.20)}$$

where $C_{ki}$, $K_{ki}$ and $D_{ki}$ are the effective stiffnesses of a layered shell. When the reference surface of the shell coincides with the inner surface of the layered shell, they have the expressions as

$$C_{ki} = \sum_{j=1}^{N} B_{ki}^{j}(H_j - H_{j-1}) \tag{E.21}$$

$$K_{ki} = \frac{1}{2} \sum_{j=1}^{N} B_{ki}^{j}(H_j^2 - H_{j-1}^2)$$

$$D_{ki} = \frac{1}{3} \sum_{j=1}^{N} B_{ki}^{j}(H_j^3 - H_{j-1}^3)$$

where $B_{ki}$ has been defined in Appendix A, and $H_j$ is the distance measured from the reference surface to the jth contact surface. For a single-layered orthotropic shell, the middle surface of the shell is usually chosen to be a reference surface. In this case, the stiffness $K_{ki}$ is equal to zero.

Equations (E.19) and (E.20) may be combined into following two differential equations through successive differentiation and elimination:

$$\frac{d^6W}{dx^6} + 3s_2 \frac{d^4W}{dx^4} + 3s_1 \frac{d^2W}{dx^2} + s_0W = 0 \tag{E.22}$$

and

$$U = m_1 \frac{dW}{dx} + m_2 \frac{d^3W}{dx^3} + m_3 \frac{d^5W}{dx^5} \tag{E.23}$$

where

$$s_0 = M_2/m_3M_1,\ s_1 = (M_3 + m_1M_1)/(3m_3M_1)\ , \tag{E.24}$$

$$s_2 = (M_4 + M_1m_2)/(3m_3M_1)\ ,$$

and where

$$m_1 = (C_{11}M_2 - \frac{C_{12}}{R} M_1)/m_o\ , \tag{E.25}$$

$$m_2 = (C_{11}M_3 + K_{11}M_1)/m_o\ ,$$

$$m_3 = C_{11}M_4/m_o\ ,$$

$$m_0 = B_{11}M_1h\ \bar{\lambda}^2\ ,$$

$$M_1 = \frac{C_{11}C_{12}}{R} + B_{11}K_{11}h\ \bar{\lambda}^2\ ,$$

$$M_2 = \frac{C_{11}C_{22}}{R^2} - C_{11}B_{11}h\ \bar{\lambda}^2\ ,$$

$$M_3 = \frac{C_{12}K_{11}}{R} + \frac{2C_{11}K_{12}}{R}\ ,$$

$$M_4 = C_{11}D_{11} - K_{11}^2\ .$$

Using the same general procedures presented in Chapter III, the frequencies of single-layered shell made of topaz with $R = 6$ inches and $R/h = 100$ for various $\ell/R$ are obtained and shown in Figure 4.

# LITERATURE CITED


1. Shtaerman, I. Ia., "On the Theory of Symmetrical Deformation of Anisotropic Elastic Shells," Izv. Kievsk. i Sel-Khoz. Institute, 1924.

2. Ambartsumian, S. A., Theory of Anisotropic Shells, NASA Technical Translation F-118, May 1964.

3. Dong, S. B., "Membrane Stresses in Laminated Anisotropic Shells," EM3, Journal of Engineering Mechanics, Part 1, June 1964, pp. 53-59.

4. Mushtari, Kh. M., "Some Generalizations on the Theory of Thin Shells with Application to the Solution of Stability of Elastic Anisotropic Shells," Izv. Kazanskogo Fiz.-Mat. Obshchestva, 11, Kazan, 1938.

5. Mushtari, Kh. M., "Some Generalizations on the Theory of Thin Shells," Prikl. Mat. Mekh. 2, no. 4, 1939.

6. Ambartsumian, S. A., "On the Problem of the Calculation of Laminated Anisotropic Shells," Izvestiia Akademii Nauk, Armyanskoi, S.S.R., Vol. 6, no. 3, 1953 (in Russian).

7. Eason, G., "On the Vibration of Anisotropic Cylinders and Spheres," Applied Science Research (A), 1963, pp. 81-85.

8. Das, Y. C., "Vibrations of Orthotropic Cylindrical Shells," Appl. Sci. Res. (A), 1964, pp. 317-327.

9. Mirsky, I., "Three-Dimensional and Shell Theory Analysis for Axisymmetric Vibration of Orthotropic Shells," Journal of Acoustic Society of American, 39, 3, March 1966, pp. 549-555.

10. Mirsky, I., "Axisymmetric Vibrations of Orthotropic Cylinders," J. Acoust. Sco. Amer. 35, 11, Nov. 1964, pp. 2106-2112.

11. Kalnins, A., "On Free and Force Vibration of Rotationally Symmetric Layered Shells," Journal of Applied Mechanics, Brief Note, Dec. 1965, pp. 941-943.

12. Ahmed, N., "Axisymmetric Plane-Strain Vibrations of a Thick Layered Orthotropic Cylindrical Shells," J. Acoust. Soc. Amer. Dec. 1966, pp. 1509-1516.

13. Ambartsumian, S. A., "Contributions to the Theory of Anisotropic Layered Shells," J. Appl. Mech. Riews, Vol. 15, no. 4, April 1962.

14. Boal, J. L. and Reissner, E., "Three-Dimensional Theory of Elastic Plates with Transverse Inextensibility," Journal of Mathematics and Physics, Vol. XXXIX, 1960, pp. 161-181.

15. Handbook of Chinese Engineers, Fundamental, Mar. 1963 (in Chinese).

16. Trent, H. M. and Stone, D. E., American Institute of Physics Handbook, D. E. Gray et al., Eds., McGraw-Hill Book Co., 1963, Chapter 2, p. 58.

17. Clark, S. P., Handbook of Physical Constants, Revised Edition, Geological Society of America, Inc., Table 5-2.

18. Dong, S. B. and Pister, K. S. and Taylor, R. R., "On the Theory of Laminated Anisotropic Shells and Plates," Journal of Aerospace Science, Aug. 1962, pp. 969-975.

19. Reissner, E., "On the Form of Variationally Derived Shell Equations," J. Appl. Mech., June 1964, pp. 233-238.

20. Naghdi, P. M., "On a Variational Theorem in Elasticity and Its Application to Shell Theory," J. Appl. Mech., Dec. 1964, pp. 647-653.

21. Love, A.E.H., A Treatise on the Mathematical Theory of Elasticity, Fourth Edition, Dover Publications, N. Y., 1944.

22. Habip, L. M., "A Review of Recent Work on Multilayered Structures," International Journal of Mechanical Science, Vol. 7, 1965, pp. 589-593.

23. Hearman, R. F. S., An Introduction to Applied Anisotropic Elasticity, Oxford Univ. Press, 1961.

24. Stravsky, Y., "Bending and Stretching of Laminated Aeolotropic Plates," Proceedings of the American Society of Civil Engineers, Journal of Engineering Mechanics Division, Vol. 87, EM6, Dec. 1961, pp. 31-56.

25. Novozhilov, V. V., Thin Shell Theory, Revised 2nd Edition, P. Noordhiff Ltd., Groningen-The Netherlands, 1964.

26. Paul, B., "Linear Bending Theory of Laminated Cylindrical Shells Under Axisymmetric Load," J. Appl. Mech., Vol. 30, 1963, pp. 98-102.

27. Bert, C. W., "Structural Theory for Laminated Anisotropic Elastic Shells," Journal of Composite Materials, Vol. 1, 1967, pp. 414-423.

28. Mattheison, R. B., Pister, K. S. and Taylor, R. L., "Analysis of Structural Laminates," Report No. ARL 76, Aeronautical Research Lab., Office of Aerospace Research, USAF, Wright-Patterson AFB, September 1961.

29. Spollers, W. R., "A Laminated Thin Cylindrical Shell Under Axisymmetric Static Loading," Int. J. Mech. Sci. Vol. 9, 1967.

30. Gulati, S. T. and Essenburg, F., "Effects of Anisotropy in Axisymmetric Cylindrical Shells," J. Appl. Mech., September 1967, pp. 659-661.

31. Timoshenko, S., Theory of Plates and Shells, McGraw-Hill, New York, 1959.

32. Bauer, H. F., Hsu, T. M. and Wang, J. T. S., "Interaction of a Sloshing Liquid with Elastic Containers," Journal of Basic Engineering, Transactions of the ASME, September 1968, pp. 373-377.

33. Bauer, H. F., Hsu, T. M. and Wang, J. T. S., "Liquid Sloshing in Elastic Containers," NASA CR-882, September 1967.

34. Kraus, H., Thin Elastic Shells, John Wiley and Sons, Inc., N. Y., 1967.

35. Forsberg, K., "Influence of Boundary Conditions on the Modal Characteristics of Thin Cylindrical Shells," AIAA Journal , December, 1964, pp. 2150-2157.

36. Forsberg, K., "A Review of Analytical Methods Used to Determine the Modal Characteristics of Cylindrical Shells," NASA CR-613, September 1966.

37. Greenspon, J. E., "Vibrations of a Thick-Walled Cylindrical Shell--Comparison of the Exact Theory with Approximate Theories," J. Acoust. Soc. Amer. 32, 5, May 1960, pp. 571-578.

# VITA

Teh-Min Hsu was born in Tainan, Taiwan, China on October 3, 1940. He received the Bachelor of Science degree in Civil Engineering from Cheng-Kung University, Taiwan, China in June 1963. In September 1964, he attended the Department of Civil Engineering at the University of Arizona, Tucson, Arizona, from which he received the Master of Science degree in the field of engineering mechanics in February, 1966.

From March 1966 to June 1968, he worked as a graduate research assistant in the School of Engineering Science and Mechanics. Since June 1968, he has been working as an aircraft structures engineer at Lockheed-Georgia Aircraft Company.